Data-driven Delay Estimation in Reaction-Diffusion Systems via Exponential Fitting

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Abstract: For a reaction-diffusion equation with unknown right-hand side and non-local measurements subject to unknown constant measurement delay, we consider the nonlinear inverse problem of estimating the associated leading eigenvalues and measurement delay from a finite number of noisy measurements. We propose a reconstruction criterion and, for small enough noise intensity, prove existence and uniqueness of the desired approximation and derive closed-form expressions for the first-order condition numbers, as well as bounds for their asymptotic behavior in a regime when the number of measurements tends to infinity and the inter-sampling interval length is fixed. We perform numerical simulations indicating that the exponential fitting algorithm ESPRIT is first-order optimal, namely, its first-order condition numbers have the same asymptotic behavior as the analytic ones in this regime.

Keywords: Time-delay systems, Data-driven control, Estimation

1. INTRODUCTION

Reaction-diffusion equations (RDEs) are widely used to model phenomena in physics and engineering, including magnetized plasma, flame front propagation and chemical processes (Sivashinsky 1977; Nicolaenko 1986). RDEs belong to the class of distributed parameters systems, and their control and observation have been extensively investigated over the last decades, see e.g. Balas (1988); Harkort and Deutscher (2011); Katz and Fridman (2022). In particular, observation and control of RDEs through modal decomposition was employed e.g. by Christofides (2001); Curtain (1982); Katz and Fridman (2021a). Almost all existing control and observation techniques assume explicit knowledge of the spatial operator of the system or of the eigenvalue/eigenfunction pairs corresponding to its modes.

Identification of unknown parameters in RDEs is a challenging problem, mostly studied in an adaptive estimation framework (Demetriou and Rosen 1994; Banks and Kunisch 2012). Adaptive estimation relies on a persistency of excitation assumption, which may be difficult to verify in practice. It also requires continuous-time measurements of the state and has not been generalized so far to a sampled-data framework and/or to estimation from a finite number of measurements. Finally, translation of these theoretical methods into tractable and efficient algorithms is, to the best of our knowledge, still an open problem. Other identification methods, accompanied by sound numerical algorithms, have been derived in the field of inverse problems (Lowe et al. 1992; Rundell and Sacks 1992; Kirsch 2011). These approaches treat the problem of recovering the spatial operator of the system under the assumption of complete knowledge of its eigenvalues. However, this assumption is non-realistic from a control theory perspective, since often only discrete-time measurements of the state are available. Hence, constructive and implementable data-driven identification techniques for reaction-diffusion equations are still missing.

In this work, we consider a 1D reaction-diffusion equation with unknown right-hand side and non-local measurements subject to an unknown, but upper bounded, constant measurement delay. Our goal is to estimate the delay and a finite number of dominant modes, corresponding to the RDE right-hand side. Our main contributions are the following:

(1) Differently from existing adaptive estimation methods, which require measurements of the form \( y(t), t \geq 0, \) or \( \{y(s_k)\}_{k=1}^{\infty} \) with \( \lim_{k \to \infty} s_k = \infty \) and inter-sampling periods of sufficiently small length, we assume that measurements are taken at finitely many uniformly distributed time steps with arbitrary inter-sampling period length. Moreover, the measurements contain structured noise, with intensity \( \varepsilon > 0 \), which emanates from measuring ‘undesirable’ system modes.

(2) We reformulate the identification task in the framework of exponential fitting, a central topic in data analysis (Istratov and Vyvenko 1999; Pereyra and Scherer 2010; Batenkov et al. 2021). We define a reconstruction criterion, and prove existence and uniqueness of the associated approximation, if the intensity \( \varepsilon \) of the structured noise is not too large (Theorem 1).

(3) For the exponential fitting problem, we introduce first-order condition numbers (Theorem 1), which describe how the \( \varepsilon \)-noise is amplified in the reconstruction errors, and provide explicit expressions for them and for their asymptotic behavior in a specific parameter regime (see (21) and Theorem 2).

(4) We numerically compute the approximations via the ESPRIT algorithm (Roy and Kailath 1989) and show that it achieves first-order optimality, meaning that its first-order con-

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dition numbers exhibit the same asymptotic behaviour as the analytic ones, in the considered regime.

2. MODEL AND PRELIMINARIES

We consider the 1D reaction-diffusion equation

$$z(x,t) = (p(x)z(x,t))_t + q(x)z(x,t),$$

with $x \in (0,1)$, $z(x,t) \in \mathbb{R}$ and $z(0,t) = z(1,t) = 0$, subject to non-local measurements

$$y(t) = \int_0^1 c(x)z(x,t-D)dx \in \mathbb{R}, \quad t \geq 0,$$

where $c \in L^2(0,1)$ is partially known and belongs to a certain class of kernels (see Assumption 1 in Section 3). The unknown smooth functions $p, q : [0, 1] \to \mathbb{R}$ satisfy the bounds

$$0 < p(x) \leq \bar{p} < \infty, \quad q \leq q(x) \leq \bar{q}, \quad x \in [0,1), \quad (3)$$

where the constants $p, q, \overline{p}, \overline{q}$ do not need to be known. The unknown constant delay $D > 0$ satisfies $D \leq D_{\text{max}}$, with a known upper bound $D_{\text{max}}$. The initial condition $z(\cdot, 0)$ is known and belongs to a class of admissible initial conditions (see Assumption 2 in Section 3); we set $z(\cdot, 0) \equiv 0$, $t < 0$.

We denote by $\mathcal{H}_0^2(0,1)$ (resp. $\mathcal{H}_0^1(0,1)$) the Sobolev space of functions $f$ defined on $[0,1]$ that are twice (resp. once) weakly differentiable with $f'' \in L^2(0,1)$ (resp. $f' \in L^2(0,1)$ and $f(0) = f(1) = 0$). Define the operator $\varphi$ set $\pi^2/2B^2 \leq \lambda_m - \lambda_n \leq \pi^2/2B^2 + 1 \left( m^2 - n^2 \right)$. Then, for this $A_0$ we have that any $m > n \geq A_0$ satisfy

$$\pi^2/2B^2 \left( m^2 - n^2 \right) \leq \lambda_m - \lambda_n \leq \left( \pi^2/2B^2 + 1 \right) \left( m^2 - n^2 \right).$$

Choosing $u_0 = \pi^2/2B^2$ and $Y_0 = \pi^2/2B^2 + 1$, we obtain (6).

Next, consider $m, n \in [A_0 - 1]$. Since $[A_0 - 1]$ is a finite set, there exist some $u_0 \leq u_0$ and $Y_0 < Y_1$ such that

$$u_0 \left( m^2 - n^2 \right) \leq \lambda_m - \lambda_n \leq Y_0 \left( m^2 - n^2 \right), \quad n \leq m \leq A_0 - 1. \quad (7)$$

In particular, (6) continues to hold with $u_0, Y_0$ replaced by $u_1, Y_1$, respectively. To finish the proof, we now show that there exist $u \leq Y_1$ and $Y_1 < Y$ such that

$$u \left( m^2 - n^2 \right) \leq \lambda_m - \lambda_n \leq Y \left( m^2 - n^2 \right), \quad n \leq A_0 - 1, \quad m \geq A_0. \quad (8)$$

Assume $u$ cannot be found such that the lower bound in (8) holds. Then, setting $u = 2 - q$, $q \in \mathbb{N}$, there exist $n_q \leq A_0 - 1$, $m_q \geq A_0$ such that

$$2 - q \geq \lambda_{m_q} - \lambda_{n_q} \geq \pi^2 \left( m_q^2 - n_q^2 \right) + q - \bar{q} / m_q^2 - 1. \quad (4)$$

Taking $q \to \infty$, which implies $m_q \to \infty$, we have $0 \geq \pi^2 p > 0$, which is a contradiction. Similar arguments hold for $Y$.

3. PROBLEM STATEMENT AND ASSUMPTIONS

Employing modal decomposition (Katz and Fridman 2021a), we present the solution to system (1) as

$$z(x,t) = \sum_{n=1}^{\infty} z_n(t) \psi_n(x), \quad z_n(t) = (z(\cdot,t), \psi_n), \quad n \in \mathbb{N}. \quad (9)$$

Differentiating under the integral sign and integrating by parts, we have

$$z_n(t) = -\lambda_n z_n(t) \Rightarrow z_n(t) = e^{-\lambda_n t} z_n(0) \text{ for all } n \in \mathbb{N}, \text{ whence}$$

$$z(x,t) = \sum_{n=1}^{\infty} z_n(0) e^{-\lambda_n t} \psi_n(x). \quad (10)$$

Substituting (10) into (2), for $t \geq D_{\text{max}}$ we obtain

$$y(t) = \sum_{n=1}^{\infty} c_n z_n(0) e^{-\lambda_n t} \psi_n(x), \quad c_n = \langle c, \psi_n \rangle, \quad n \in \mathbb{N}. \quad (11)$$

Before formally stating our identification objective, we state our main assumptions on the system (1) and the measurements (2). There exist $N_1, N_2 \in \mathbb{N}$ such that the following properties hold.

Assumption 1. The measurement kernel $c \in L^2(0,1)$ belongs to the class of kernels whose coefficients $\{c_n\}_{n=1}^{\infty}$ satisfy

(a) $c_n = 0$ for all $n > N_1 + N_2$,
(b) $c_n$ are known and nonzero for $n \in [N_1]$, 
(c) $c_k \in [N_1 + N_2]$ for some $M_k > 0$ $e^{ck} \leq M_k \{c_n\}_{n=1}^{\infty}$ for all $n \in [N_1]$ and all $k \in [N_1 + N_2] \setminus [N_1]$. 

Assumptions 1(a) and 1(b) mean that $c \in L^2(0,1)$ is a bandlimited measurement kernel, supported on $\{\psi_{n=1}^{N_1+N_2}\}$, with known and nonzero projection coefficients on $\{\psi_{n=1}^{N_1}\}$. In addition, Assumption 1(c) means that the projection coefficients on $\{\psi_{n=1}^{N_1+N_2}\}$ are “small” in comparison to those on $\{\psi_{n=1}^{N_1}\}$. In signal processing terms, this means that the measurement kernel $c$ (whose eigenstructure is shown in Figure 1) has a main lobe on the frequency domain $\{\lambda_{n=1}^{N_1}\}$ as well as an undesirable small side lobe on the frequency domain $\{\lambda_{n=1}^{N_1}\}$, which is associated with structured noise in the measurements.
Fig. 1. Eigenstructure of the kernel $c \in L^2(0,1)$.

Assumption 2. The initial condition $z(0) \in L^2(0,1)$, $z_n(0) \neq 0$, is known for $n \in [N_1]$. Furthermore, $\frac{z_n(0)}{z_n(\Delta)} \leq M_z$ for some $M_z > 0$, $n \in [N_1]$ and $k \in [N_1 + N_2] \setminus [N_1]$.

Assumption 3. The system measurements are taken only at times $t_k := k\Delta + D_{\max} \frac{2^{N_1-1}}{k-0}$, with step-size $\Delta > 0$.

Assumption 3 implies that we only have finitely many “snapshots” of the system output.

Subject to Assumptions 1-3, the measurements (11) at the available times $\{t_k\}_{k=0}^{2^{N_1}-1}$ can be presented as

$$y(t_k) = \sum_{n=1}^{N_1} y_n e^{-\lambda_n t_k} \Delta + \varepsilon \sum_{n=N_1+1}^{N_1+N_2} y_n e^{-\lambda_n t_k} \Delta,$$

for $k = 0, \ldots, 2N_1 - 1$, where

$$y_n := \begin{cases} e^{c_n z_n(0)} e^{\lambda_n (D-D_{\max})}, & n \in [N_1] \\ e^{c_n z_n(0)} e^{\lambda_n (D-D_{\max})}, & n \in [N_1+N_2] \setminus [N_1] \end{cases},$$

satisfy $|y_n| \leq M_zM_\varepsilon := M_y$ for all $n \in [N_1], k \in [N_1+N_2] \setminus [N_1]$, since $e^{(\lambda_n - \epsilon) t_k} (D-D_{\max}) < 1$ for such indices, as $D < D_{\max}$.

Identification objective: Given the measurements (12) and $N_0 \in [N_1]$, estimate the eigenvalues $\{\lambda_n\}_{n=1}^{N_0}$ and the constant delay $D$.

The problem of recovering $\{y_n, \lambda_n\}_{n=1}^{N_1}$ from the measurements (12) is known as exponential fitting (Perera and Scherer 2010). The components $\{y_n(t_k)\}_{k=0}^{2^{N_1}-1}$ in the measurements constitute an “$e$-structured” measurement noise, emanating from the fact that $c$ is not a perfect filter (i.e., $c \notin \text{span}\{y_n(t_k)\}_{k=1}^{N_0}$). For $\varepsilon = 0$, there exist multiple methods which recover $\{y_n, \lambda_n\}_{n=1}^{N_1}$ exactly; see the discussion of the ESPRIT algorithm (Roy and Kailath 1989) in Section 5. However, if $\varepsilon > 0$, the structured measurement noise introduces errors into the estimation. To the best of our knowledge, error estimates for exponential fitting in the presence of small structured noise do not exist currently in the literature (except in some special cases such as Batenkov et al. 2021). The goal of this work is to study the analytic estimation errors due to noise, to first order in $\varepsilon$, and to show that the ESPRIT algorithm is first-order optimal in achieving the identification objective, thereby gaining insight into the system (1).

The considered problem is highly challenging for two reasons. First, we assume that only finitely many measurements are available for the reconstruction procedure, for any triplet $(\Delta, N_1, N_2)$. Second, although (1) is a linear system, the task of recovering $\{D, \lambda_n\}_{n=1}^{N_0}$ from the measurements (12) is a nonlinear inverse problem, as the measurements depend nonlinearly on these parameters.

4. IDENTIFICATION CRITERION AND ITS ANALYSIS

Given (12), we introduce the map

$$F(\{\hat{y}_n, \hat{\Delta}_n\}_{n=1}^{N_1}) := \text{col} \left\{ \sum_{n=1}^{N_1} \hat{y}_n e^{-\lambda_n \hat{\Delta}_n} - y(t_k) \right\}_{k=0}^{2^{N_1}-1}.$$  \hspace{1cm} (14)

Given an approximation candidate $\hat{P}$, the function $F(\hat{P})$ returns the discrepancy between measurements $\{y(t_k)\}_{k=0}^{2^{N_1}-1}$ and “virtual measurements” $\left\{ \sum_{n=1}^{N_1} \hat{y}_n e^{-\lambda_n \hat{\Delta}_n} \right\}_{k=0}^{2^{N_1}-1}$.

In particular, if $\varepsilon = 0$ (i.e., there is no structured noise in the measurements), we see that $F(\hat{P})$ is well defined, we prove the existence and uniqueness of a $\varepsilon$-approximation of $\hat{P}$ if $\hat{P}(\varepsilon) = 0$. To avoid ambiguity, we always assume that the elements of $\hat{P}$ are sorted in increasing order of eigenvalues, i.e. $\hat{\lambda}_k < \hat{\lambda}_{k+1}$ for all $k$.

Given an $\varepsilon$-approximation $\hat{P}$, one can generate a $\phi$-approximation $\hat{D}$ by

$$\hat{D}(\varepsilon) = \min_{\hat{P}} F(\hat{P}) \quad \text{subject to } \hat{P}(\varepsilon) = 0.$$

Remark 1. In (13) $\hat{\lambda}_n$ and $y_n$, $n \in [N_1]$, are not independent. However, in (14) we search for a candidate $\hat{P}$, where $\hat{\lambda}_n$ and $\hat{y}_n$, $n \in [N_1]$, are treated as independent. This approach may lead to a loss of structure, but it has the key advantage of yielding a well-posed inverse problem (see Theorem 1) for whose solution tractable numerical algorithms exist (see Section 5).

In the following analysis, only to keep the presentation simpler, we assume that $N_2 = 1$: the sum $y_{\text{mail}}(t_k), k \in [0] \cup [2N_1 - 1]$, in (12) contains a single term. Our analysis and conclusions remain identical for an arbitrary fixed $N_2 \in \mathbb{N}$.

Hereafter, we use the notation

$$\phi_n := e^{-\lambda_n \Delta}, \quad \hat{\phi}_n := e^{-\hat{\lambda}_n \Delta}, \quad n \in [N_1 + 1]. \hspace{1.5cm} (15)$$

The measurements in (12) are then rewritten as

$$y(t_k) = \sum_{n=1}^{N_1} y_n \phi_n + \varepsilon y_{N_1+1} \phi_{N_1+1}, k \in [0] \cup [2N_1 - 1]. \hspace{1cm} (16)$$

To show that our criterion is well defined, we prove the existence and uniqueness of an $\varepsilon$-approximation, for small $\varepsilon > 0$.

Theorem 1. There exist $\varepsilon_0 > 0$ and unique continuously differentiable functions $\hat{P}(\varepsilon) := \{\hat{y}_n(\varepsilon), \hat{\Delta}_n(\varepsilon)\}$ such that $\hat{P}(0) = \{y_n, \lambda_n\}_{n=1}^{N_1}$ and for all $|\varepsilon| < \varepsilon_0$, $F(\hat{P}(\varepsilon)) = 0 \iff \hat{P}(\varepsilon)$.

Furthermore, the components of $\hat{P}(\varepsilon)$ are continuously differentiable on $|\varepsilon| < \varepsilon_0$ and satisfy as $\varepsilon \to 0$

$$\hat{\lambda}_n(\varepsilon) - \lambda_n = \mathcal{K}_n(n;N_1, \Delta) e + o_n N_1 \Delta(\varepsilon), \quad \hat{y}_n(\varepsilon) - y_n = \mathcal{K}_n(n;N_1, \Delta) e + o_n N_1 \Delta(\varepsilon), \hspace{1cm} (17)$$

for

$$\mathcal{K}_n(n;N_1, \Delta) = \begin{cases} \frac{H_{\varepsilon,n}(\phi_{N_1+1})}{\Delta \varepsilon \phi_n} & n \in [N_1] \end{cases}, \hspace{1cm} (18)$$

Here $\{H_{\varepsilon,n}, \phi_{\varepsilon,n}\}_{n=1}^{N_1}$ are the Hermite interpolation basis polynomials, given in (A.2), associated with $\Phi = \{\phi_n\}_{n=1}^{N_1}$. \hfill $\diamond$
The terms $\mathcal{X}_\epsilon(n;N_1,\Delta)$ and $\mathcal{Y}_\epsilon(n;N_1,\Delta)$, $n \in [N_1]$ are the first order (in $\epsilon$) condition numbers of the problem. Henceforth, we will suppress their dependence on $N_1,\Delta$ for brevity.

**Proof.** $\mathcal{F}(\hat{P},\epsilon)$ is differentiable in all variables $(\hat{P},\epsilon)$. We denote by $\partial_{\hat{P}}\mathcal{F}(\hat{P},0)$ its Jacobian with respect to $\hat{P}$ evaluated at $\hat{P} = [y_n,\lambda_n]_{n=1}^{N_1} =: P$ and $\epsilon = 0$. A direct computation yields

$$
\partial_{\hat{P}}\mathcal{F}(P,0) = \text{diag} \left\{ \left[ \frac{1}{0 - \lambda_n \phi_n} \right]_{n=1}^{N_1} \right\},
$$

(19)

where $W_\Phi$ is, which is associated with $\Phi$, is given in (A.3).

Since the eigenvalues $\lambda_n$ are simple, it follows from the uniqueness of Hermite interpolation that $W_\Phi$ is invertible and $W_\Phi^{-1} = \mathbb{H}_\Phi(\Phi)$. In view of Assumptions 1-2 and of (13), we have that $\gamma_n \neq 0$, $n \in [N_1]$, whereas $\theta_n \neq 0$, $n \in [N_1]$, by definition. Therefore, $\lim_{\|e\| \to 0} \partial_{\hat{P}}\mathcal{F}(P,0) \neq 0$. The implicit function theorem (Spivak 1987) guarantees that there exist $\epsilon_* > 0$ and unique continuously differentiable functions $P(\epsilon)$ such that $P(0) = P$ and $\mathcal{F}(P(\epsilon),\epsilon) = 0$ for all $\epsilon < \epsilon_*$. Differentiating $\mathcal{F}(\hat{P},\epsilon)$ with respect to $\epsilon$ and substituting $\epsilon = 0$, we obtain

$$
\frac{\partial}{\partial \epsilon} \mathcal{F}(P(\epsilon),\epsilon) |_{\epsilon = 0} = \text{diag} \left\{ \left[ \frac{1}{0 - \lambda_n \phi_n} \right]_{n=1}^{N_1} \right\} \mathbb{H}_\Phi(\Phi) \text{col} \left\{ \phi_n \right\}_{n=1}^{N_1}.
$$

(20)

Since (A.4) holds, we obtain the expression in (18).

**Remark 1.** Recalling the Lagrange polynomials given in (A.1) we observe that $L_{\Phi,n}^2(\phi_{n+1};\epsilon) = \frac{\xi_1}{\xi_2}$ and $|L_{\Phi,n}^2(\phi_n)| = \xi_3$. \(\diamond\)

To prove our main result on the asymptotic behavior of the condition numbers, we need several lemmas. In the next, we employ the notations in Proposition 2 in the Appendix.

**Lemma 1.** The functions in (22) can be written as

$$
\xi_1 = e^{-2\Delta\psi_{2,1}(y_n - \lambda_n)} \phi_n, \quad \xi_2 = e^{-2\Delta\psi_{2,1}(y_n - \lambda_n)} \phi_n, \quad n > N_1,
$$

$$
\xi_3 = e^{-2\Delta\psi_{2,1}(y_n - \lambda_n)} \phi_n, \quad n < N_1,
$$

where

$$
\theta_1 := \sum_{j=1}^{N_1} \log \left(1 - e^{-\Delta\psi_{2,1}(\phi_n - \phi_j)}\right) > 0,
$$

$$
\theta_2 := \sum_{j=1}^{N_1} \log \left(1 - e^{-\Delta\psi_{2,1}(\phi_n - \phi_j)}\right) > 0, \quad n > 1,
$$

$$
\theta_3 := \sum_{j=1}^{N_1} \log \left(1 - e^{-\Delta\psi_{2,1}(\phi_n - \phi_j)}\right) > 0, \quad n < N_1
$$

satisfy the inequalities

$$
\theta_1 \leq F_0(\psi_{2,1}(\phi_n + N_1)) + \log \left(1 - e^{-\Delta\psi_{2,1}(\phi_n + N_1)}\right),
$$

$$
\theta_1 \geq F_{j_2}(\Delta\psi_{2,1}(y_n + 1)) + \log \left(1 - e^{-\Delta\psi_{2,1}(y_n + 1)}\right) + 2\Delta\psi_{2,1}(y_n + 1) + \log \left(1 - e^{-\Delta\psi_{2,1}(y_n + 1)}\right),
$$

(23)

$$
\theta_2 \geq F_{j_2}(\Delta\psi_{2,1}(y_n - 1)) + \log \left(1 - e^{-\Delta\psi_{2,1}(y_n - 1)}\right) - 2\Delta\psi_{2,1}(y_n - 1) + \log \left(1 - e^{-\Delta\psi_{2,1}(y_n - 1)}\right),
$$

(24)

where the positive constants $\psi$ and $\Delta\psi$ are those given in Proposition 1 and the function $\mathcal{F}_{w_1,w_2}$ is given in (B.1). \(\diamond\)

**Proof.** We consider $\xi_1$ only. The results for $\xi_2$ and $\xi_3$ are proved similarly. We have

$$
\log |\xi_1| = -2\Delta \sum_{j=1}^{N_1} \lambda_j - 2\theta_{2,1},
$$

with $\theta_1$ given in (23). Employing (5) in (23), we obtain

$$
\theta_1 \geq \sum_{j=1}^{N_1} \log \left(1 - e^{-\Delta\psi_{2,1}(\phi_n - \phi_j)}\right).
$$

Let $\ell := \log \left(1 - e^{-\Delta\psi_{2,1}(\phi_n + N_1)}\right)$. Then, we have

$$
\sum_{j=1}^{N_1} \log \left(1 - e^{-\Delta\psi_{2,1}(\phi_n - \phi_j)}\right) - \ell \geq \sum_{j=1}^{N_1} \log \left(1 - e^{-\Delta\psi_{2,1}(\phi_n + N_1)}\right)
$$

$$
\geq \int_{0}^{\Delta\psi_{2,1}(\phi_n + N_1)} f_{\Delta\psi_{2,1}(\phi_n + N_1)}(x) dx = \mathcal{F}_{j_2}(\Delta\psi_{2,1}(\phi_n + N_1)) \geq F_{j_2}(\Delta\psi_{2,1}(\phi_n + 1)).
$$

(25)

The first inequality holds as $(N_1 + 1)^2 - j^2 < (N_1 + 1 - j)(N_2 + 1)$, while the second one holds because the sum in the second row can be viewed as Riemannian sum of the positive and monotonically decreasing function $\mathcal{E}(\Delta\psi_{2,1}(x))$ over $x \in [1, N_1]$. Hence, the integral provides a lower bound for the sum. The upper bound is proved analogously, using $(N_1 + 1 - j)(N_1 + 1) \leq (N_1 + 1)^2 - j^2$, $\theta_1 \leq \sum_{j=1}^{N_1} \log \left(1 - e^{-\Delta\psi_{2,1}(\phi_n - \phi_j)}\right)$ and

$$
\sum_{j=1}^{N_1} \log \left(1 - e^{-\Delta\psi_{2,1}(\phi_n + j)}\right)
$$

$$
\geq \int_{0}^{\Delta\psi_{2,1}(\phi_n + 1)} f_{\Delta\psi_{2,1}(\phi_n + 1)}(x) dx = \mathcal{F}_{j_2}(\Delta\psi_{2,1}(\phi_n + 1)) \geq F_{j_2}(\Delta\psi_{2,1}(\phi_n + 1)).
$$

Next, recall the Lagrange polynomials $\{L_{\Phi,n}\}_{n=1}^{N_1}$ as in (A.1), where $\Phi = \{\phi_n\}_{n=1}^{N_1}$.

**Lemma 2.** For $n \in [N_1]$, we have

$$
L_{\Phi,n}^2(\Phi_{n+1}) = \left\{ \begin{array}{ll}
\frac{\xi_1}{\xi_2} + \frac{\xi_1}{\xi_2} \Theta, & n < N_1, \\
\frac{\xi_1}{\xi_2} + \frac{\xi_1}{\xi_2} \Theta, & n = N_1,
\end{array} \right.
$$

(26)

where

$$
\Theta = \left\{ \begin{array}{ll}
-2(\theta_1 - \theta_2 - \theta_3), & n > 1, \\
-2(\theta_1 - \theta_2), & n = 1.
\end{array} \right.
$$

(25)
Moreover, fixing \( n \in [N_1 - 1] \), \( \Delta > 0 \) and denoting
\[
\sigma^{N_1}_n := \frac{N_1(N_1 + 1)(2N_1 + 1)}{6} - n(n + 1)(2n + 1) - (N_1 - n)^2,
\]
there exists a constant \( M_\phi = M_\phi(\Delta) > 0 \) such that
\[
L_{\psi, n}^2(\phi_{N_1 + 1}) \leq M_\phi e^{-2\Delta \sigma^{N_1}_n} = M_\phi e^{\frac{2\Delta \sigma^{N_1}_n}{3}} (1 + O_n(N_1^{-2})) .
\]  
\( \diamond \) (27)

**Proof.** The equality (25) follows from Lemma 1 and the fact that \( L_{\psi, n}^2(\phi_{N_1 + 1}) = \frac{\xi}{N_1^3} \).

Fix \( n \in [N_1 - 1] \) and consider (25). By Proposition 1,
\[
-2\Delta \sum_{j=k+1}^{N_1} (\lambda_j - \lambda_k)^5 \leq -2\Delta \sigma^{N_1}_n = \frac{2\Delta \sigma^{N_1}_n}{3} (1 + O_n(N_1^{-2})). \quad (28)
\]
as \( N_1 \to \infty \) (note that \( O_n(N_1^{-2}) \) is independent of \( \Delta \)). On the other hand, consider the lower and upper bounds in (24). In view of these bounds and Proposition 2, \( \theta_1, \theta_2; \theta_3 \) and \( \theta_4 \) are uniformly bounded in \( N_1 \gneq n \) (recall that \( \lambda \) and \( n \) are considered fixed). Hence, \( \Theta \) in (26) is uniformly bounded for all \( N_1 \in \mathbb{N} \). Combining the latter with (25) and (28), we obtain (27).

**Lemma 3.** The term \( \xi_n \) in (22) satisfies
\[
\frac{\xi_n}{\phi_n} \leq M_\xi \frac{e^{\Delta \lambda_n}}{\Delta} \quad (29)
\]
for some \( M_\xi > 0 \) independent of \( \Delta > 0 \). \( \diamond \)

**Proof.** We write \( \xi_n = \xi_{n, 1} + \xi_{n, 2} \), where
\[
\xi_{n, 1} = \sum_{k=[n-1]}^{N_1} \frac{1}{\phi_n - \phi_k} \quad \text{and} \quad \xi_{n, 2} = \sum_{k=n+1}^{N_1} \frac{1}{\phi_n - \phi_k} .
\]
For \( \xi_{n, 1} \) with \( n > 1 \), we have
\[
\xi_{n, 1} \leq \frac{e^{\Delta \lambda_n}}{\Delta} \sum_{k=1}^{N_1} \frac{1}{\lambda_k - \lambda_n} \leq \frac{e^{\Delta \lambda_n}}{\Delta} \frac{1}{(N_1 - n + 1)} \sum_{k=1}^{N_1} \frac{1}{\lambda_k - \lambda_n} \leq \frac{e^{\Delta \lambda_n}}{\Delta} \frac{1 + \ln(2n + 1)}{2n} .
\]  
where the first inequality follows from the application of Lagrange’s theorem with the derivative computed at \( \lambda_n \), the second follows from (5). The third inequality follows from comparison with the integral of the positive and monotonically increasing function \( x \mapsto (n^2 - x^2)^{-1} \) on \( x \in [1, n - 1] \). Analogously, for \( n < N_1 \) we obtain
\[
\xi_{n, 2} \leq \Delta \sum_{k=1}^{N_1} \frac{1}{\lambda_k - \lambda_n} \leq \Delta \sum_{k=1}^{n} \frac{1}{\lambda_k - \lambda_n} \leq \Delta \sum_{k=1}^{n} \frac{1}{\lambda_k - \lambda_n} \leq \frac{e^{\Delta \lambda_n}}{\Delta} \frac{1 + \ln(2n + 1)}{2n} .
\]
The result follows from (30) and (31), since (31) is satisfied.

We can now establish the asymptotic behaviour of the condition numbers \( \mathcal{K}_n(\lambda) \) and \( \mathcal{K}_n(\Lambda) \) in the regime (21).

**Theorem 2.** Recall the first-order condition numbers \( \mathcal{K}_n(\lambda) \) and \( \mathcal{K}_n(\Lambda) \) in (17). Let \( n \in \mathbb{N} \). Given \( \Delta > 0 \), there exist some \( \gamma_\lambda(n, \Delta) > 0 \) and \( \gamma_\Lambda(n, \Delta) > 0 \) such that, as \( N_1 \to \infty \),
\[
|\mathcal{K}_n(n)| \leq \gamma_\lambda(n, \Delta) \left| y_{N_1 + 1}(0) \right| \xi_{n, 1} | \mathcal{K}_n(n)| \leq \gamma_\Lambda(n, \Delta) e^{-\frac{2\Delta \sigma^{N_1}_n}{3} (1 + O(N_1^{-2}))} ,
\]
\( \diamond \)

The condition numbers \( \mathcal{K}_n(\lambda) \) and \( \mathcal{K}_n(\Lambda) \) determine, to first order in \( \epsilon \), how much an \( \epsilon \)-perturbation in the measurements is amplified in the \( \epsilon \)-approximation \( \tilde{P} \); in view of Theorem 2, in the regime (21) they decay super-exponentially with \( N_1 \to \infty \).

**Proof.** For \( \mathcal{K}_n(\lambda) \), in view of (18) and (A.2), we have
\[
|\mathcal{K}_n(\lambda)| \leq |y_{N_1 + 1}(1 + 2\epsilon - \Delta \xi_n)| L_{\psi, n}^2(\phi_{N_1 + 1}) .
\]
Employing (27) and (29),
\[
|\mathcal{K}_n(\lambda)| \leq M_{\phi} |y_{N_1 + 1}(1 + 2\epsilon - \Delta \xi_n)| L_{\psi, n}^2(\phi_{N_1 + 1}) .
\]
Recalling (13), (28) and using the fact that, when \( N_1 \to \infty \),
\[
|\mathcal{K}_n(\Lambda)| = M_{\phi} \left( \frac{1 + 2\epsilon - \Delta \xi_n}{\phi_{N_1 + 1}} \right) L_{\psi, n}^2(\phi_{N_1 + 1}) .
\]
By Assumptions 1-3, \( \frac{|y_{N_1 + 1}|}{\phi_{N_1 + 1} - \phi_n} \leq M_\theta \), whereas \( \frac{|\phi_{N_1 + 1} - \phi_n|}{\phi_{N_1 + 1}} \leq 1 \). Hence, from (27), we again have
\[
|\mathcal{K}_n(\Lambda)| \leq M_{\phi} M_\phi \Delta^{-1} e^{-2\Delta \sigma^{N_1}_n} .
\]
In light of (28), we obtain the bound on \( \mathcal{K}_n(\lambda) \). \( \diamond \)

**Remark 3.** Theorem 2 continues to hold uniformly for \( n \leq \beta N_1 \), \( \beta < 1 \) as \( N_1 \to \infty \). In this case, \( \sigma^{N_1}_n \) is lower bounded by a positive constant for all \( n \leq \beta N_1 \), thereby the super-exponential decay rate is preserved.

5. NUMERICAL RESULTS

Our numerical simulations, implemented in Wolfram Mathematica, show that the first-order condition numbers of the ESPRIT algorithm (Roy and Kailath 1989) exhibit the same asymptotic behaviour as \( \mathcal{K}_n(\lambda) \) and \( \mathcal{K}_n(\Lambda) \) in the regime (21).

5.1 Multi-exponential model with structured perturbations

We start by examining the numerical conditioning of the multi-exponential model (12) with a structured (multi-exponential) perturbation term. These simulations are aimed at verifying the behavior of the analytic condition numbers of the multi-exponential model (12). A complete PDE delay estimation will be presented in the next subsection. We set \( \lambda_n = n^2, \gamma_n = 1 \) for all \( n \), and fix \( \Delta = 0.04, N_1 = 1 \). For various values of \( N_1 \), we compute the ideal condition numbers \( \mathcal{K}_n(\lambda) \) and \( \mathcal{K}_n(\Lambda) \), \( n \in [N_1] \), given by (18). The results are shown in Fig. 2a. Super-exponential decay is clearly seen, as predicted by Theorem 2.

The ESPRIT algorithm (Roy and Kailath 1989) is one of the best-performing methods for exponential fitting. It requires at least \( 2N_1 \) equispaced samples of the signal \( y(t) \) of the form (12), and produces estimates of the parameters \( \{y_n, \lambda_n\}_{n=1}^{N_1} \). It provides exact solutions when \( \epsilon = 0 \), and performs close to optimal in the presence of noise, in the context of the so-called super-resolution problem in applied harmonic analysis (Batenkov et al. 2021). We apply ESPRIT to the sequence \( \{y(t)\}_{t=1}^{2N_1-1} \), with the same setup as described above. In Fig. 2b, we see that the conditioning of the ESPRIT algorithm is consistent with Theorem 2 and the computed condition numbers in Fig. 2a. We plot the rescaled errors (recall (17)),
\[
\mathcal{E}_\lambda(n) = \epsilon^{-1} \lambda_n^{\text{ESP}} - \lambda_n, \quad \mathcal{E}_\lambda(n) = \epsilon^{-1} \lambda_n^{\text{ESP}} - y_n ,
\]
where \( \lambda_n^{\text{ESP}} \) and \( y_n^{\text{ESP}} \) are the parameter values recovered by ESPRIT, and, furthermore, the \( \lambda_n^{\text{ESP}} \)’s have been sorted in increasing order. Here \( \epsilon = 10^{-3} \), and the results were computed with 100 decimal digits of precision.
Fig. 2. Condition numbers and ESPRIT performance.

5.2 PDE delay estimation

We test the complete procedure on a PDE identification problem. We consider the RDE (1) with constant \( p \equiv q \equiv 0.1 \). The eigenvalues and eigenfunctions are explicitly given by \( \lambda_n = n^2 \pi^2 - q \) and \( \psi_n(x) = \sqrt{2} \sin(n \pi x) \). The initial condition is set to satisfy \( z_n(0) = (-1)^{n+1} \sqrt{2n^3} \). To solve the RDE, we use the method of lines for space discretization with \( N \lambda \) collocation points and 32nd order finite difference approximation to attain high accuracy. The resulting ODE system is integrated for \( t \in [0, 1.5] \), with the resulting solution and the initial condition plotted in Fig. 3. Our implementation utilized the NDSolve library function.

Next, we consider the measurement model (2) with \( (a-priori \ unknown) \) delay \( D = 1/12 \). We fix the sampling step size to be \( \Delta = 1/25 \). We further fix \( D_{\text{max}} = 1/10 \), for each \( N_1 = 1, \ldots, N_{\text{max}} = 10 \). First, the measurement function is chosen with random coefficients \( c_n \in [1, 2] \) and \( \varepsilon = 0.01 \):

\[
c(x) = \sum_{n=1}^{N_1} c_n \psi_n(x) + \varepsilon \sum_{n=N_1+1}^{N_1+N_2} c_n \psi_n(x).
\]

Next, the measurement function (2) is computed by global adaptive quadrature as implemented in the NIntegrate library function, and sampled at the points \( t_k = D_{\text{max}} + k \Delta \), for \( k = 0, \ldots, 2N_1 - 1 \), giving the measurement vector \( \{y(t_k)\}_{k=0}^{2N_1-1} = \sum_{n=1}^{N_1} c_n \psi_n(0) e^{-\lambda_n (D_{\text{max}} + D) e^{-\lambda_n D}} e^{-\lambda_n \Delta} \) plus \( \varepsilon \sum_{n=N_1+1}^{N_1+N_2} c_n \psi_n(0) e^{-\lambda_n (D_{\text{max}} + D) e^{-\lambda_n D}} e^{-\lambda_n \Delta} \).

Finally, we apply the ESPRIT algorithm to the measurement vector \( \{y(t_k)\}_{k=0}^{2N_1-1} \) and recover \( \{\lambda_n^{\text{ESP}}\}_{n=1}^{N_1} \) directly. Then, since \( c_n, z_n(0), D_{\text{max}} \) are known, we can recover the approximation to \( D \) from the coefficients \( y_n^{\text{ESP}} \) as

\[
D \approx \hat{D}_{\text{ESP}}^{(1)} := \frac{1}{\lambda_1^{\text{ESP}}} \log \frac{y_1^{\text{ESP}}}{c_1 z_1(0) e^{-\lambda_1^{\text{ESP}} D_{\text{max}}}}.
\]

Whenever the argument of the logarithm is negative, we consider the reconstruction to be unsuccessful.

Fig. 3. The RDE initial condition and the solution.

The errors \( |\hat{\lambda}_1 - \lambda_1| \) and \( |\hat{D}_{\text{ESP}}^{(1)} - D| \) are shown in Fig. 4. The overall shape of the error curves is consistent with the theoretical predictions in Theorem 2 and the numerical conditioning in the previous section.

Fig. 4. Errors in \( \hat{\lambda}_1 \) and the estimated delay.

REFERENCES


### Appendix A. LAGRANGE AND HERMITIAN INTERPOLATION

Consider a set of distinct nodes $\chi = \{\chi_n\}_{n \in |S|} \subseteq \mathbb{R}$ and values $\{f_n\}_{n \in |S|} \subseteq \mathbb{R}$. The classical Lagrange interpolation problem seeks a polynomial $q(z)$, deg$(q) \leq S - 1$, satisfying $q(\chi_n) = f_n$ for all $n \in |S|$. It is well known that the solution to the problem is given by $q(z) = \sum_{n \in |S|} f_n L_{\chi_n}(z)$, where the Lagrange interpolation basis is

$$L_{\chi_n}(z) = \prod_{j \neq n} \frac{z - \chi_j}{\chi_n - \chi_j}, \quad n \in |S|.$$  \hspace{1cm} (A.1)

Given the Lagrange interpolation basis (A.1), one can construct the corresponding Hermite interpolation basis is

$$H_{\chi_n}(z) = \left[ 1 - 2(z - \chi_n) \right] L_{\chi_n}^1(z), \quad n \in |S|. \hspace{1cm} (A.2)$$

The Hermite basis is related to this interpolation problem: introducing further $\{f'_n\}_{n \in |S|}$, one seeks a polynomial $r(z)$, deg$(r) \leq 2S - 1$ satisfying $r(\chi_n) = f_n$ and $r'(\chi_n) = f'_n$ for all $n \in |S|$. The solution is given by $r(z) = \sum_{n \in |S|} (f_n H_{\chi_n}(z) + f'_n \tilde{H}_{\chi_n}(z))$.

An alternative form to the Hermite interpolation problem can be formulated as follows. Given polynomial $p(z) = \sum_{j=0}^{2S-1} a_j z^j$, we introduce the coordinate map

$$\mathcal{E}(p) = [a_0, \ldots, a_{2S-1}]^\top.$$

Then, the Hermite interpolating polynomial $r(z)$ satisfies

$$W_X \mathcal{E}(r) = \begin{cases} f_n \mathcal{E}(f_n) \end{cases}_{n=1}^S,$$

$$W_X := \begin{cases} 1 & \chi_n \cdot \chi_n \
 0 & 2 \chi_n \
 1 & (2S - 1) \chi_n \end{cases}.$$

The matrix $W_X$ maps $\mathcal{E}(r)$ to the values of $p(z)$ and $p'(z)$ at the interpolation nodes $\{\chi_n\}_{n=1}^S$. Since the Hermite interpolation problem is always solvable, $W_X$ is invertible. Moreover, it can be verified that

$$H_S(\chi) := W_X^{-1} \begin{cases} \mathcal{E}(H_{\chi_n}) \mathcal{E}(\tilde{H}_{\chi_n}) \end{cases}_{n=1}^S \in \mathbb{R}^{2S \times 2S}$$

is the unique matrix satisfying

$$H_S(\chi) \mathcal{E}(\zeta) = \begin{cases} [H_{\chi_n}(\zeta)]_{n=1}^S \end{cases} \quad (A.4)$$

for all $\zeta \in \mathbb{R}$.

### Appendix B. INTEGRAL CONVERGENCE

**Proposition 2.** Let $0 \leq w_1 < w_2 \leq \infty$ and define

$$\mathcal{Q}_\alpha(x) = -\log \left(1 - e^{-\alpha x}\right) > 0, \quad x \in (0, \infty),$$

where $\alpha > 0$. The integrals

$$\mathcal{I}_{w_1, w_2}(\alpha) := \int_{w_1}^{w_2} \mathcal{Q}_\alpha(x) \, dx \quad (B.1)$$

are finite, decreasing, and $\lim_{\alpha \to \infty} \mathcal{I}_{w_1, w_2}(\alpha) = 0$.

**Proof.** We prove the result for $w_1 = 1$ and $w_2 = \infty$ (other cases are similar). Integrating by parts, we have

$$\mathcal{I}_{1, \infty}(\alpha) = \left[-x \log \left(1 - e^{-\alpha x}\right)\right]_1^\infty + \alpha \int_1^\infty \frac{x}{1 - e^{-\alpha x}} e^{-\alpha x} \, dx.$$

The first term on the right-hand side has

$$\lim_{x \to \infty} x \log (1 - e^{-\alpha x}) = -\alpha \lim_{x \to \infty} \frac{x^2}{1 - e^{-\alpha x}} e^{-\alpha x} = 0,$$

whereas

$$0 < \int_1^\infty \frac{x}{1 - e^{-\alpha x}} e^{-\alpha x} \, dx \leq \frac{e^\alpha}{e^\alpha - 1} \int_1^\infty xe^{-\alpha x} \, dx < \infty.$$

Hence, $\mathcal{I}_{1, \infty}(\alpha) < \infty$.

Next, for a fixed $x \in (0, \infty)$,

$$0 \leq \alpha \to -\log \left(1 - e^{-\alpha x}\right) \in (0, \infty)$$

is decreasing, whence $\mathcal{I}_{1, \infty}(\alpha)$ is also decreasing. Let $\varepsilon > 0$, $\alpha > 1$ and $M > 1$. Then,

$$\mathcal{I}_{1, \alpha}(\alpha) \leq \mathcal{I}_{1, M}(\alpha) + \mathcal{I}_{M, \infty}(1) = \int_1^M -\log \left(1 - e^{-\alpha x}\right) \, dx + \int_M^\infty -\log \left(1 - e^{-\varepsilon x}\right) \, dx.$$

Choosing $M$ so that the rightmost integral is smaller than $\frac{\varepsilon}{\alpha}$ and then $\alpha$, such that $\mathcal{I}_{M}(\alpha) < \frac{\varepsilon}{\alpha}$ for $\alpha > \alpha_*$, we have that $\alpha > \min(1, \alpha_*)$ implies $0 < \mathcal{I}_{1, \infty}(\alpha) < \varepsilon$. 
