Data-driven identification of reaction-diffusion dynamics from finitely many non-local noisy measurements by exponential fitting

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***Abstract*— Given a reaction-diffusion equation with unknown right-hand side, we consider a nonlinear inverse problem of estimating the associated leading eigenvalues and initial condition modes from a finite number of non-local noisy measurements. We define a reconstruction criterion and, for a small enough noise, we prove the existence and uniqueness of the desired approximation and derive closed-form expressions for the first-order condition numbers, as well as bounds for their asymptotic behavior in a regime when the number of measured samples is fixed and the inter-sampling interval length tends to infinity. When computing the sought estimates numerically, our simulations show that the exponential fitting algorithm ESPRIT is first-order optimal, as its first-order condition numbers have the same asymptotic behavior as the analytic condition numbers in the considered regime.**

***Index Terms*— Identification, Distributed parameter systems, Data-driven control, Estimation**

I. Introduction and Considered Model

Reaction-diffusion equations (RDEs) are widely used to model phenomena in physics and engineering, including magnetized plasma, flame front propagation and chemical processes [1], [2]. RDEs belong to the class of distributed parameters systems, and their control and observation have been investigated over the last decades, see e.g. [3], [4], [5]. In particular, observation and control of RDEs through modal decomposition was employed e.g. in [6], [7], [8]. Almost all existing control and observation techniques assume explicit knowledge of the spatial operator of the system or of the eigenvalue/eigenfunction pairs corresponding to its modes.

Identification of unknown parameters in RDEs is a challenging problem, which has been mostly studied in an adaptive estimation framework [9], [10]. Adaptive estimation relies on a persistency of excitation assumption, which may be difficult to verify in practice. It also requires continuous-time measurements of the state and has not been generalized so far to a sampled-data framework and/or to estimation from a *finite* number of measurements. Finally, translation of these theoretical methods into tractable and efficient algorithms is, to the best of our knowledge, still an open problem. Other identification methods, which are accompanied by sound numerical algorithms, have been derived in the field of inverse problems [11], [12], [13]. These approaches treat the problem of recovering the spatial operator of the system under the assumption of *complete knowledge* of its eigenvalues. However, this assumption is non-realistic from a control theory perspective, since often only discrete-time measurements of the state are available. Hence, constructive and implementable data-driven identification techniques for reaction-diffusion equations are still missing.

We consider the 1D reaction-diffusion equation

$$z_t(x,t) = \left(p(x)z_x(x,t)\right)_x + q(x)z(x,t),\ z(0,t) = z(1,t) = 0, \quad (1)$$

subject to non-local measurements

$$y(t) = \int_0^1 c(x)z(x,t)dx \in \mathbb{R}, \quad t \geq 0. \quad (2)$$

Here $c \in L^2(0,1)$ is <u>partially known</u> (see Assumption 1.2 in Section III), $x \in (0,1)$ and $z(x,t) \in \mathbb{R}$. The smooth functions $p(x)$ and $q(x)$ and the initial condition $z(\cdot,0)$ are assumed <u>unknown</u>. The system (1) has an associated sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ with eigenfunctions $\{\psi_n\}_{n=1}^{\infty}$ (see Section II). The identification objective considered in this paper is the approximation of the leading eigenvalues $\{\lambda_n\}_{n=1}^{N_0}$, $N_0 \in \mathbb{N}$, and the initial condition modes from the available measurements, subject to appropriate assumptions (see Section III). The contribution of the paper is as follows:

1) Differently from existing adaptive estimation algorithms, which require measurements of the form $y(t)$, $t \geq 0$, or $\{y(s_k)\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} s_k = \infty$, we assume that the measurements (2) are available at <u>finitely many</u> time steps. Moreover, c in (2) is not a <u>perfect filter</u> (meaning, $c \notin \operatorname{span}\{\psi_n\}_{n=1}^{N_1}$) and introduces structured noise into the measurements, with intensity ε, which emanates from measuring 'undesirable' system modes.
2) We define a reconstruction criterion in the presence of structured noise, and prove the existence and uniqueness of the associated approximation, provided $\varepsilon > 0$ is not too large (see Theorem 1).
3) We introduce first-order condition numbers, in (16), which describe how the ε-noise is amplified in the reconstruction errors, and provide explicit expressions for these condition numbers, as well their asymptotic behavior in the specific regime described in (17).
4) Finally, we consider the problem of numerically computing the approximations. The parameter identification problem turns out to be a special case of *exponential fitting*, a classical topic in data analysis with

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numerous applications [14], [15], [16]. Our numerical simulations show that the well-known ESPRIT algorithm [17] achieves first-order optimality, meaning that the first-order condition numbers of the ESPRIT algorithm exhibit the same asymptotic behaviour as the analytic condition numbers, in the considered regime.

We believe that our results pave the way towards new directions in data-driven identification of RDEs.

II. Preliminaries

Consider the system (1), where the *unknown* smooth functions satisfy the bounds

$$0 < \underline{p} \le p(x) \le \overline{p} < \infty, \ \underline{q} \le q(x) \le \overline{q}, \quad x \in [0,1]. \tag{3}$$

The constants $\underline{p}, \underline{q}, \overline{p}, \overline{q}$ are assumed to be *unknown*. We denote by $\mathscr{H}^2(0,1)$ (resp. $\mathscr{H}^1_0(0,1)$) the Sobolev space of functions f defined on $[0,1]$ that are twice (resp. once) weakly differentiable with $f'' \in L^2(0,1)$ (resp. $f' \in L^2(0,1)$ and $f(0)=f(1)=0$). Define the operator $\mathscr{A}$

$$\begin{aligned}&[\mathscr{A}h](x) = -(p(x)h'(x))' - q(x)h(x), \quad x \in (0,1),\\ &\mathrm{Dom}(\mathscr{A}) = \left\{h \in \mathscr{H}^2(0,1) \ ; \ h(0)=h(1)=0\right\}.\end{aligned}$$

The operator $\mathscr{A}$ has an infinite monotone sequence of simple eigenvalues $\{\lambda_n\}_{n=1}^\infty$ satisfying $\lim_{n\to\infty}\lambda_n = \infty$ [18]. The eigenvectors $\{\psi_n\}_{n=1}^\infty$ form a complete orthonormal system in $L^2(0,1)$. Also, they satisfy the inequalities [18]

$$\pi^2 n^2 \underline{p} + \underline{q} \le \lambda_n \le \pi^2 n^2 \overline{p} + \overline{q}, \quad n \in \mathbb{N}. \tag{4}$$

Well-posedness of system (1) has been studied thoroughly [19]. In particular, given $z(\cdot,0) \in L^2(0,1)$, system (1) has a unique solution $z \in C([0,\infty), \mathscr{H}^1_0(0,1)) \cap C^1((0,\infty), \mathscr{H}^1_0(0,1))$ such that $z(\cdot,t) \in \mathrm{Dom}(\mathscr{A})$ for all $t > 0$. We present the solution to (1) as

$$z(x,t) = \sum_{n=1}^\infty z_n(t)\psi_n(x), \quad z_n(t) = \langle z(\cdot,t), \psi_n\rangle, \ n \in \mathbb{N}. \tag{5}$$

Differentiating under the integral sign and integrating by parts, we have $\dot{z}_n(t) = -\lambda_n z_n(t)$ for all $n \in \mathbb{N}$, whence

$$z(x,t) = \sum_{n=1}^\infty z_n(0) e^{-\lambda_n t}\psi_n(x). \tag{6}$$

Henceforth, we adopt the notation $[n] = \{i \in \mathbb{N} \ ; \ 1 \le i \le n\}$.

Proposition 1: There exist constants $\upsilon, \Upsilon > 0$ such that

$$\upsilon\left(m^2 - n^2\right) \le \lambda_m - \lambda_n \le \Upsilon\left(m^2 - n^2\right) \tag{7}$$

holds for every choice of $1 \le n < m$.

Proof: We first show that (7) holds for $A \le n < m$, with some $A \in \mathbb{N}$. By [20, Equation 4.21], the eigenvalues $\{\lambda_n\}_{n=1}^\infty$ have the asymptotic behavior $\lambda_n = \frac{\pi^2}{B^2}n^2 + a_0 + O\left(\frac{1}{n^2}\right)$, $n \ge 1$, where B and a_0 are *positive* constants. Hence, $\lambda_m - \lambda_n \ge \frac{\pi^2}{2B^2}\left(m^2 - n^2\right) + \frac{\pi^2}{2B^2} + O\left(\frac{1}{A^2}\right)$, which implies the lower bound in (7), for large enough A. The upper bound is proved via similar arguments. Next, given A, it is clear that by increasing Υ and decreasing ν, we can further guarantee (7) for $1 \le n < m \le A$. We show that Υ and υ can be tuned such that (7) holds for $1 \le n \le A$ and $m > A$. Assume ν cannot be found such that (7) holds. Then, for any $q \in \mathbb{N}$, there exist $n_q \le A$, $m_q > A$ such that

$$2^{-q} > \frac{\lambda_{m_q} - \lambda_{n_q}}{m_q^2 - n_q^2} \overset{(4)}{\ge} \frac{\pi^2\left(\underline{p}m_q^2 - \overline{p}A^2\right) + \underline{q} - \overline{q}}{m_q^2 - 1}.$$

By $\lim_{q\to\infty} m_q = \infty$ we get a contradiction. Similar arguments hold for Υ. ∎

For $\chi = \{\chi_n\}_{n=1}^S \subseteq \mathbb{C}$, the Lagrange interpolation basis is

$$L_{\chi,n}(z) = \prod_{j\ne n} \frac{z - \chi_j}{\chi_n - \chi_j}, \quad n \in [S], \tag{8}$$

and the corresponding Hermite interpolation basis is

$$\begin{aligned}H_{\chi,n}(z) &= \left[1 - 2(z-\chi_n)L'_{\chi,n}(\chi_n)\right]L^2_{\chi,n}(z),\\ \widetilde{H}_{\chi,n}(z) &= (z-\chi_n)L^2_{\chi,n}(z), \quad n \in [S].\end{aligned} \tag{9}$$

The Hermite basis contains polynomials of degree at most $2S-1$. For a polynomial $q(z) = \sum_{j=0}^S a_j z^j$, we introduce the coordinate map $\mathfrak{C}(q) = \mathrm{col}\left\{a_j\right\}_{j=0}^S$. The Hermite matrix $\mathbb{H}_S(\chi) = \left(\mathrm{row}\left\{\left[\mathfrak{C}(H_{\chi,n}) \quad \mathfrak{C}(\tilde{H}_{\chi,n})\right]\right\}_{n=1}^S\right)^\top \in \mathbb{R}^{2S\times 2S}$ is the unique matrix satisfying

$$\mathbb{H}_S(\chi)\,\mathrm{col}\left\{\zeta^j\right\}_{j=0}^{2S-1} = \mathrm{col}\left\{\begin{bmatrix}H_{\chi,n}(\zeta)\\ \widetilde{H}_{\chi,n}(\zeta)\end{bmatrix}\right\}_{n=1}^S. \tag{10}$$

Proposition 2: Let $0 \le w_1 < w_2 \le \infty$ and $\mathscr{Q}_\alpha(x) = -\log\left(1 - e^{-\alpha x}\right)$ with $x > 0$ and $\alpha > 0$. The integrals

$$\mathscr{J}_{w_1,w_2}(\alpha) := \int_{w_1}^{w_2} \mathscr{Q}_\alpha(x)\mathrm{d}x > 0 \tag{11}$$

are finite and decreasing, and $\lim_{\alpha\to\infty} \mathscr{J}_{w_1,w_2}(\alpha) = 0$.

Proof: We prove the result for $w_1 = 1$ and $w_2 = \infty$; other cases are similar. Integrating by parts, we have

$$\mathscr{J}_{1,\infty}(\alpha) = \left[-x\log\left(1 - e^{-\alpha x}\right)\right]_1^\infty + \alpha\int_1^\infty \frac{x}{1 - e^{-\alpha x}}e^{-\alpha x}\mathrm{d}x.$$

Since $\lim_{x\to\infty} x\log\left(1 - e^{-\alpha x}\right) = 0$ and the integral on the right-hand side converges, we have that $\mathscr{J}_{1,\infty}(\alpha) < \infty$. Given $x \in (0,\infty)$, $(0,\infty) \ni \alpha \mapsto -\log\left(1 - e^{-\alpha x}\right) \in (0,\infty)$ is decreasing, whence $\mathscr{J}_{1,\infty}(\alpha)$ is decreasing too. Let $\varepsilon > 0$, $\alpha > 1$ and $M > 1$. Then, $\mathscr{J}_{1,\infty}(\alpha) \le \mathscr{J}_{1,M}(\alpha) + \mathscr{J}_{M,\infty}(1)$. Choosing M so that the rightmost term is smaller than $\frac{\varepsilon}{2}$ and then, by Dini's theorem, α_* such that $\mathscr{J}_{1,M}(\alpha) < \frac{\varepsilon}{2}$ for $\alpha > \alpha_*$, we have that $\alpha > \min(1, \alpha_*)$ implies $0 < \mathscr{J}_{1,\infty}(\alpha) < \varepsilon$. ∎

III. The Identification Objective

A. Measurements and standing assumptions

Considering system (1) subject to the non-local measurements (2), we substitute (6) into (2) to obtain

$$y(t) = \sum_{n=1}^\infty c_n z_n(0)e^{-\lambda_n t}, \quad c_n = \langle c, \psi_n\rangle, \ n \in \mathbb{N}. \tag{12}$$

Remark 1: We *do not assume* that the solution to (1) is exponentially stable. In fact, for $|\overline{q}| > 0$ large, the solution (6) may contain (finitely) many unstable modes.

Before formally stating our identification objective, we present our main assumptions on system (1) and measurements (2). Let there exist $N_1, N_2 \in \mathbb{N}$ such that the following properties hold.

Assumption 1: The coefficients $\{c_n\}_{n=1}^\infty$ satisfy

1) $c_n = 0$ for all $n > N_1 + N_2$,
2) c_n are *known and nonzero* for $n \in [N_1]$,
3) $\frac{|c_k|}{|c_n|} \le M_c \varepsilon$ for all $n \in [N_1], k \in [N_1+N_2] \setminus [N_1]$

for some $\varepsilon > 0$ and $M_c > 0$. We define

$$\tilde{c}_k := \frac{c_k}{\varepsilon},\ k \in [N_1+N_2] \setminus [N_1].$$

Assumption 2: The initial condition $z(\cdot,0) \in L^2(0,1)$ is *unknown* and satisfies $z_n(0) \neq 0$ for $n \in [N_1]$. Furthermore, $\frac{|z_k(0)|}{|z_n(0)|} \le M_z$ for some $M_z > 0$, $n \in [N_1]$ and $k \in [N_1+N_2] \setminus [N_1]$.

Remark 2: Our proposed approach allows to approximate $\{y_n\}_{n=1}^{N_1}$ even if both $\{c_n\}_{n=1}^{N_1}$ and $\{z_n(0)\}_{n=1}^{N_1}$ are unknown. However, knowledge of $\{y_n\}_{n=1}^{N_1}$ in (14) does not allow to separate $\{c_n\}_{n=1}^{N_1}$ and $\{z_n(0)\}_{n=1}^{N_1}$ from their (corresponding) products, if both sets are unknown. This is an inherent ambiguity of the identification objective. In Assumptions 1 and 2, we consider $\{c_n\}_{n=1}^{N_1}$ to be known and $\{z_n(0)\}_{n=1}^{N_1}$ to be unknown. Our approach is also valid for unknown $\{c_n\}_{n=1}^{N_1}$ and known $\{z_n(0)\}_{n=1}^{N_1}$.

Assumption 3: The system measurements are available *only at a finite set of times* $\{t_k := k\Delta\}_{k=0}^{2N_1-1}$, with step-size $\Delta > 0$.

Subject to Assumptions 1-3, the measurements (12) at the available times $\{t_k\}_{k=0}^{2N_1-1}$ can be presented as

$$y(t_k) = \underbrace{\sum_{n=1}^{N_1} y_n e^{-\lambda_n \Delta k}}_{y_{\text{main}}(t_k)} + \varepsilon \underbrace{\sum_{n=N_1+1}^{N_1+N_2} y_n e^{-\lambda_n \Delta k}}_{y_{\text{tail}}(t_k)},\ k = 0,\ldots,2N_1-1 \quad (13)$$

where

$$y_n := \begin{cases} c_n z_n(0), & n \in [N_1] \\ \tilde{c}_n z_n(0), & n \in [N_1+N_2] \setminus [N_1] \end{cases} \quad (14)$$

satisfy $\frac{|y_k|}{|y_n|} \le M_c M_z =: M_y$ for all $n \in [N_1], k \in [N_1+N_2] \setminus [N_1]$.

B. Identification objective and considered regime

We use the measurements (13) to estimate $\{z_n(0), \lambda_n\}_{n=1}^{N_0}$, for some $N_0 \le N_1$. To define a recovery criteria, we introduce

$$\mathscr{F}\left(\left\{\hat{y}_n, \hat{\lambda}_n\right\}_{n=1}^{N_1}, \varepsilon\right) = \operatorname{col}\left\{\sum_{n=1}^{N_1} \hat{y}_n e^{-\hat{\lambda}_n \Delta k} - y(t_k)\right\}_{k=0}^{2N_1-1}, \quad (15)$$

which reflects the discrepancy between the measurements $\{y(\Delta k)\}$ and the "virtual measurements" $\left\{\sum_{n=1}^{N_1} \hat{y}_n e^{-\hat{\lambda}_n \Delta k}\right\}$ obtained from a candidate $\hat{P} := \left\{\hat{y}_n, \hat{\lambda}_n\right\}$. Note that $y(t_k)$ in (13) depends on ε.

Definition 1: $\hat{P}$ is an *ε-approximation* of $\{y_n, \lambda_n\}$ if $\mathscr{F}\left(\hat{P}; \varepsilon\right) = 0$.

The intuition behind Definition 1 is as follows:

- Measurements (13) are split into y_{main} and y_{tail}. The former contains the desired $\{z_n(0), \lambda_n\}_{n=1}^{N_0}$, whereas the latter is an *ε-structured perturbation*, where $\varepsilon > 0$, since $c \in L^2(0,1)$ is not a perfect filter, i.e. $c \notin \operatorname{span}\{\psi_n\}_{n=1}^{N_1}$.
- When c is a perfect filter, $\varepsilon = 0$, and thus $\mathscr{F}(P;0) = 0$, where $P = \{y_n, \lambda_n\}$. In particular, the ε-approximation coincides with the true parameter, namely $P = \hat{P}$.
- Hence, we propose to seek an approximation $\hat{P}$ which preserves the equality $\mathscr{F}\left(\hat{P}; \varepsilon\right) = 0$ for $\varepsilon > 0$.

Given an ε-approximation $\hat{P}$, Assumption 1 and expression (14) allow us to derive estimates for the dominant $N_0 \le N_1$ projection coefficients of the initial condition $z(\cdot,0)$, $\{z_n(0)\}_{n=1}^{N_0}$, since $\hat{z}_n(0) = \frac{\hat{y}_n}{c_n}$, $\quad n \in [N_0]$.

We address the analytical problem of existence of $\hat{P} = \hat{P}(\varepsilon)$. Specifically:

1) Given a small $\varepsilon > 0$, we show that there exists a unique $\hat{P} = \hat{P}(\varepsilon)$ and we derive the first-order expansions of the reconstruction errors with respect to ε:

$$\begin{aligned} \frac{e_\lambda(n)}{\varepsilon} &:= \frac{\hat{\lambda}_n(\varepsilon) - \lambda_n}{\varepsilon} = \mathscr{K}_\lambda(n) + o(1), \\ \frac{e_y(n)}{\varepsilon} &:= \frac{\hat{y}_n(\varepsilon) - y_n}{\varepsilon} = \mathscr{K}_y(n) + o(1),\ n \in [N_0],\ \varepsilon \to 0, \end{aligned} \quad (16)$$

where $\mathscr{K}_\lambda(n)$ and $\mathscr{K}_y(n)$ are the *first-order condition numbers* for the recovery of $\hat{P}$, which describe how much the ε-structured perturbation is amplified when estimating the reconstruction errors $e_\lambda(n)$ and $e_y(n)$.

2) Assuming N_2 fixed, we derive the asymptotics of $\mathscr{K}_\lambda(n)$ and $\mathscr{K}_y(n)$ for $n \in [N_1]$ in the

$$\underline{\text{Regime:}}\ N_1 \text{ fixed and } \Delta \to \infty \quad (17)$$

in order to obtain insight into the behaviour of the reconstruction errors $e_\lambda(n)$ and $e_y(n)$ for small ε.

3) When computing $\hat{P}$, our numerical simulations show that the ESPRIT algorithm [17] is first-order optimal, i.e., its first-order condition numbers exhibit the same asymptotic behaviour as $\mathscr{K}_\lambda(n)$ and $\mathscr{K}_y(n)$ in the regime (17).

Remark 3: The considered problem is highly challenging for two reasons. First, we assume that only *finitely many* measurements are available for the reconstruction procedure, for any triplet (Δ, N_1, N_2). Second, although (1) is a linear system, the task of recovering $\{y_n, \lambda_n\}_{n=1}^{N_0}$ from the measurements (13) is a *nonlinear inverse problem*, as the measurements depend *nonlinearly* on these parameters.

IV. Identification Problem: First-Order Analysis

Here we derive explicit expressions for the first-order condition numbers $\mathscr{K}_\lambda(n)$ and $\mathscr{K}_y(n)$ in (16), and bound their asymptotic behavior in the regime (17).

Only to keep the presentation simpler, we assume that $N_2 = 1$ (i.e., the sum $y_{\text{tail}}(t_k)$, $k \in \{0\} \cup [2N_1-1]$, in (13) contains a single term). The analysis and the conclusions of this section remain identical for an arbitrary fixed $N_2 \in \mathbb{N}$.

Throughout the section, we use the notation

$$\phi_n := e^{-\lambda_n \Delta},\ n \in [N_1+1]. \quad (18)$$

The measurements in (13) are then rewritten as

$$y(t_k) = \sum_{n=1}^{N_1} y_n \phi_n^k + \varepsilon y_{N_1+1} \phi_{N_1+1}^k,\ k \in \{0\} \cup [2N_1-1]. \quad (19)$$

We now prove the existence of an ε-approximation $\hat{P}(\varepsilon)$ and derive closed-form expressions for $\mathscr{K}_\lambda(n)$ and $\mathscr{K}_y(n)$.

Theorem 1: There exist $\varepsilon_* > 0$ and continuously differentiable functions $\hat{P}(\varepsilon) := \left\{\hat{y}_n(\varepsilon), \hat{\lambda}_n(\varepsilon)\right\}$ such that $\hat{P}(0) = P$ and that, for all $|\varepsilon| < \varepsilon_*$, $\mathscr{F}(\hat{P}; \varepsilon) = 0 \iff \hat{P} = \hat{P}(\varepsilon)$. Furthermore, we have

$$\begin{bmatrix} \mathscr{K}_y(n) \\ \mathscr{K}_\lambda(n) \end{bmatrix} = y_{N_1+1} \begin{bmatrix} H_{\Phi,n}(\phi_{N_1+1}) \\ -\frac{1}{\Delta y_n \phi_n} \tilde{H}_{\Phi,n}(\phi_{N_1+1}) \end{bmatrix},\ n \in [N_1], \quad (20)$$

where $\left\{H_{\Phi,n},\tilde{H}_{\Phi,n}\right\}_{n=1}^{N_1}$ are the Hermite interpolation basis polynomials, given in (9), associated with $\Phi=\{\phi_n\}_{n=1}^{N_1}$.

Proof: Consider the function $\mathscr{F}(\hat{P},\varepsilon)$ in (15) and recall that $P=\{y_n,\lambda_n\}$ are the true parameters. It can be seen that $\mathscr{F}$ is differentiable in all variables $(\hat{P},\varepsilon)$. We denote by $\partial_{\hat{P}}\mathscr{F}(P,0)$ its Jacobian with respect to $\hat{P}$ evaluated at $\hat{P}=P$ and $\varepsilon=0$. Then, $\partial_{\hat{P}}\mathscr{F}(P,0)=J(P,0)D(P,0)$, with

$$J(P,0)=\mathbb{H}_{N_1}(\Phi)^{-1},\quad D(P,0)=\operatorname{diag}\left\{\begin{bmatrix}1&0\\0&-\Delta y_n\phi_n\end{bmatrix}\right\}_{n=1}^{N_1},\tag{21}$$

where $\mathbb{H}_{N_1}(\Phi)$ is the Hermite matrix, given in (10), associated with Φ. Since the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ are simple, it follows from the uniqueness of Hermite interpolation that $\mathbb{H}_{N_1}(\Phi)$ is invertible. In view of Assumptions 1-2 and of (14), we have that $y_n\neq 0,\ n\in[N_1]$, whence $\det(\partial_{\hat{P}}\mathscr{F}(P,0))\neq 0$. The implicit function theorem [21] guarantees that there exist $\varepsilon_*>0$ and continuously differentiable functions $\hat{P}(\varepsilon)$ such that $\hat{P}(0)=P$ and $\mathscr{F}(\hat{P};\varepsilon)=0\iff\hat{P}=\hat{P}(\varepsilon)$ for all $|\varepsilon|<\varepsilon_*$. Differentiating $\mathscr{F}(\hat{P}(\varepsilon),\varepsilon)=0$ with respect to ε and substituting $\varepsilon=0$, we obtain

$$\operatorname{col}\left\{\begin{bmatrix}\mathscr{K}_y(n)\\\mathscr{K}_\lambda(n)\end{bmatrix}\right\}_{n=1}^{N_1}=\partial_{\hat{P}}\mathscr{F}(P,0)^{-1}y_{N_1+1}\operatorname{col}\left\{\phi_{N_1+1}^k\right\}_{k=0}^{2N_1-1}.\tag{22}$$

Recalling that $\partial_{\hat{P}}\mathscr{F}(P,0)=\mathbb{H}_{N_1}(\Phi)^{-1}D(P,0)$ and that (10) holds, we obtain the expression in (20). ■

Theorem 1 allows us to analyze the asymptotic behaviour of the condition numbers $\left\{\mathscr{K}_y(n)\right\}_{n=1}^{N_1}$ and $\left\{\mathscr{K}_\lambda(n)\right\}_{n=1}^{N_1}$ in the regime (17) through the analysis of the polynomials in (20). Given $\Delta>0$, $N_1\in\mathbb{N}$ and $n\in[N_1]$, set the functions

$$\begin{aligned}&\xi_1=\textstyle\prod_{j\neq n}(\phi_{N_1+1}-\phi_j)^2,\quad \xi_2=\prod_{j=1}^{n-1}(\phi_n-\phi_j)^2\\&\xi_3=\textstyle\prod_{j=n+1}^{N_1}(\phi_n-\phi_j)^2,\quad \xi_4=\sum_{k\neq n}|\phi_n-\phi_k|^{-1},\end{aligned}\tag{23}$$

where all summations/products range over indices in $[N_1]$, and we use the convention that $\prod_{j=l}^{k}b_j=1$ and $\sum_{j=l}^{k}b_j=0$ whenever $k<l$. We omit the dependence of functions on (n,N_1,Δ) for simplicity of notation.

Remark 4: Recalling the Lagrange polynomials given in (8) we observe that $L_{\Phi,n}^2(\phi_{N_1+1})=\frac{\xi_1}{\xi_2\xi_3}$ and $\left|L_{\Phi,n}'(\phi_n)\right|=\xi_4$.

To prove our main result, we need several lemmas.

Lemma 1: The functions in (23) can be written as

$$\begin{aligned}&\xi_1=e^{-2\Delta\sum_{j\neq n}\lambda_j-2\theta_1},\quad \xi_2=e^{-2\Delta\sum_{j=1}^{n-1}\lambda_j-2\theta_2},\quad n>1,\\&\xi_3=e^{-2\Delta(N_1-n)\lambda_n-2\theta_3},\ n<N_1,\end{aligned}$$

where

$$\begin{aligned}&\theta_1:=\textstyle\sum_{j\neq n}-\log\left(1-e^{-\Delta(\lambda_{N_1+1}-\lambda_j)}\right)>0,\\&\theta_2:=\textstyle\sum_{j=1}^{n-1}-\log\left(1-e^{-\Delta(\lambda_n-\lambda_j)}\right)>0,\quad n>1,\\&\theta_3:=\textstyle\sum_{j=n+1}^{N_1}-\log\left(1-e^{-\Delta(\lambda_j-\lambda_n)}\right)>0,\ n<N_1\end{aligned}$$

satisfy the inequalities

$$\begin{aligned}&\theta_1\leq\mathscr{J}_{0,\infty}(\Delta\upsilon N_1)+\log\left(1-e^{-\Delta\upsilon(N_1+1-n)(N_1+1)}\right),\\&\theta_1\geq\mathscr{J}_{1,2}(\Delta\Upsilon(2N_1+1))+\log\left(1-e^{-\Delta\Upsilon(N_1+1-n)(2N_1+1)}\right),\\&\mathscr{J}_{1,2}(\Delta\Upsilon(2n-1))\leq\theta_2\leq\mathscr{J}_{0,\infty}(\Delta\upsilon(n+1)),\\&\mathscr{J}_{1,2}(\Delta\Upsilon(N_1+n+1))\leq\theta_3\leq\mathscr{J}_{0,\infty}(\Delta\upsilon(2n+1)),\end{aligned}$$

where the positive constants υ and Υ are those given in Proposition 1 and the function $\mathscr{J}_{w_1,w_2}$ is given in (11).

Proof: Due to space constraints, we consider ξ_1. The results for ξ_2 and ξ_3 are proved similarly. We have

$$\log(\xi_1)=-2\Delta\sum_{j\neq n}\lambda_j-2\theta_1.$$

Employing (7), we obtain

$$\theta_1\geq\textstyle\sum_{j\neq n}-\log\left(1-e^{-\Delta\Upsilon((N_1+1)^2-j^2)}\right).$$

Let $\ell:=\log\left(1-e^{-\Delta\Upsilon(N_1+1-n)(2N_1+1)}\right)$. Then, we have

$$\begin{aligned}&\textstyle\sum_{j\neq n}-\log\left(1-e^{-\Delta\Upsilon((N_1+1)^2-j^2)}\right)-\ell\\&\geq\textstyle\sum_{j=1}^{N_1}-\log\left(1-e^{-\Delta\Upsilon j(2N_1+1)}\right)\geq\int_1^{N_1+1}\mathscr{Q}_{\Delta\Upsilon(2N_1+1)}(x)\mathrm{d}x\\&=\mathscr{J}_{1,N_1+1}(\Delta\Upsilon(2N_1+1))\geq\mathscr{J}_{1,2}(\Delta\Upsilon(2N_1+1)),\end{aligned}$$

where the first inequality holds because $(N_1+1)^2-j^2\leq(N_1+1-j)(2N_1+1)$ and the second holds because the sum in the second row can be viewed as Riemannian sum of the positive and monotonically decreasing function $\mathscr{Q}_{\Delta\Upsilon(2N_1+1)}(x)$ over $x\in[1,N_1]$. Hence, the integral provides a lower bound for the sum. The upper bound is proved analogously, using $(N_1+1-j)(N_1+1)\leq(N_1+1)^2-j^2$. ■

Consider the Lagrange polynomials $\left\{L_{\Phi,n}\right\}_{n\in[N_1]}$ as in (8).

Lemma 2: For $n\in[N_1]$, we have

$$L_{\Phi,n}^2(\phi_{N_1+1})=\begin{cases}e^{-2\Delta\sum_{j=n+1}^{N_1}(\lambda_j-\lambda_n)+\Theta},\ n<N_1,\\e^{2\theta_2-2\theta_1},\ n=N_1,\end{cases}\tag{24}$$

where

$$\Theta=\begin{cases}-2\left(\theta_1-\theta_2-\theta_3\right),\quad n>1,\\-2\left(\theta_1-\theta_3\right),\quad n=1.\end{cases}\tag{25}$$

Moreover, fixing $n\in[N_1-1]$, $\underline{\Delta}>0$ and denoting

$$\sigma(n,N_1):=\frac{N_1(N_1+1)(2N_1+1)}{6}-\frac{n(n+1)(2n+1)}{6}-(N_1-n)n^2,$$

there exists a constant $M_\phi=M_\phi(\underline{\Delta})>0$ such that

$$L_{\Phi,n}^2(\phi_{N_1+1})\leq M_\phi e^{-2\Delta\sigma(n,N_1)},\quad\Delta\geq\underline{\Delta}.\tag{26}$$

Proof: The equality (24) follows from Lemma 1 and the fact that $L_{\Phi,n}^2(\phi_{N_1+1})=\frac{\xi_1}{\xi_2\cdot\xi_3}$. Now let $n\in[N_1-1]$. In view of Lemma 1, Θ in (25) is uniformly bounded in $\Delta\in[\underline{\Delta},\infty)$. Moreover,

$$-2\Delta\textstyle\sum_{j=n+1}^{N_1}(\lambda_j-\lambda_n)\overset{(7)}{\leq}-2\Delta\upsilon\sigma(n,N_1)\leq-2\Delta\upsilon(2N_1-1),\tag{27}$$

whereas for $\Delta\geq\underline{\Delta}$, we employ Lemma 1 and Proposition 2 to obtain

$$e^{\Theta}\leq e^{2(\mathscr{J}_{0,\infty}(2\upsilon\underline{\Delta})+\mathscr{J}_{0,\infty}(3\upsilon\underline{\Delta}))}=:M_\phi\tag{28}$$

for $n\in[N_1-1]$. (26) follows from (24),(27) and (28). ■

Proposition 3: The term ξ_4 in (23) satisfies

$$\xi_4\leq M_\xi\frac{e^{\Delta\lambda_{N_1}}}{\Delta}\tag{29}$$

for some $M_\xi>0$ independent of $\Delta>0$.

Proof: We write $\xi_4=\xi_{4,1}+\xi_{4,2}$, where

$$\xi_{4,1}=\textstyle\sum_{k\in[n-1]}\frac{1}{|\phi_n-\phi_k|}\text{ and }\xi_{4,2}=\sum_{k=n+1}^{N_1}\frac{1}{|\phi_n-\phi_k|}.$$

For $\xi_{4,1}$ with $n>1$, we have

$$\xi_{4,1}\leq\frac{e^{\Delta\lambda_n}}{\Delta}\textstyle\sum_{k=1}^{n-1}\frac{1}{\lambda_n-\lambda_k}\leq\frac{e^{\Delta\lambda_n}}{\Delta\upsilon}\sum_{k=1}^{n-1}\frac{1}{n^2-k^2}\leq\frac{e^{\Delta\lambda_n}}{\Delta\upsilon}\frac{\ln(2n)}{2n}.\tag{30}$$

where the first inequality follows from the application of Lagrange's theorem with the derivative computed at λ_n, the second follows from (7). The third inequality follows from comparison with the integral of the positive and monotonically increasing function $x \mapsto (n^2-x^2)^{-1}$ on $x \in [1, n-1]$. Analogously, for $n < N_1$ we obtain

$$\xi_{4,2} \le \Delta^{-1} \sum_{k=n+1}^{N_1} \frac{e^{\Delta\lambda_k}}{\lambda_k-\lambda_n} \overset{(7)}{\le} \frac{e^{\Delta\lambda_{N_1}}}{\Delta\upsilon} \sum_{k=n+1}^{N_1} \frac{1}{k^2-n^2} \le \frac{e^{\Delta\lambda_{N_1}}}{\Delta\upsilon} \frac{1+\ln(2n+1)}{2n}. \tag{31}$$

The result follows from (30), (31) since $\frac{\ln(2n)}{2n}$ and $\frac{1+\ln(2n+1)}{2n}$ are bounded sequences. ■

We can now establish the asymptotic behaviour of the condition numbers $\mathcal{K}_\lambda(n)$ and $\mathcal{K}_y(n)$ in the regime (17).

Theorem 2: Recall the first-order condition numbers $\mathcal{K}_\lambda(n)$ and $\mathcal{K}_y(n)$ in (16). Let $n \in \mathbb{N}$ and $\rho > 0$. There exists $N_1^*(\rho) \in \mathbb{N}$ with $n < N_1^*(\rho)$ such that, for all $N_1 > N_1^*(\rho)$,

$$\begin{aligned} |\mathcal{K}_y(n)| &\le \zeta_y(n, N_1, y_{N_1+1}) \Delta^{-1} e^{-\rho\Delta}, \\ |\mathcal{K}_\lambda(n)| &\le \zeta_\lambda \Delta^{-1} e^{-\rho\Delta}, \end{aligned} \qquad \Delta \to \infty \tag{32}$$

with $\zeta_y(\lambda_{N_1}, y_{N_1+1}) > 0$, $\zeta_\lambda > 0$ independent of Δ.

Proof: Given $\rho > 0$, let $N_1^*(\rho) > n$ be large enough such that for all $N_1 > N_1^*(\rho)$, $\lambda_{N_1} > 0$ and

$$-2\upsilon\sigma(n, N_1) + \lambda_{N_1} < -\rho. \tag{33}$$

Note that such $N_1^*(\rho) > 0$ exists since $\sigma(n, N_1)$ grows as N_1^3, whereas by (4), we have $\lambda_{N_1} = O(N_1^2)$ as $N_1 \to \infty$. Consider first $\mathcal{K}_y(n)$. In view of (20), we have

$$|\mathcal{K}_y(n)| \overset{\text{Remark 4}}{\le} |y_{N_1+1}| \left(1 + 2e^{-\Delta\lambda_n}\xi_4\right) L^2_{\phi,n}(\phi_{N_1+1}).$$

Employing (26), (29) and (33) with $\Delta \ge \underline{\Delta} > 0$,

$$\begin{aligned} |\mathcal{K}_y(n)| &\le M_\phi \, |y_{N_1+1}| \left(\Delta + 2e^{-\Delta\lambda_n} M_\xi e^{\Delta\lambda_{N_1}}\right) \Delta^{-1} e^{-2\Delta\upsilon\sigma(n,N_1)} \\ &\le M_\phi \, |y_{N_1+1}| \left(\Delta e^{-\Delta\lambda_{N_1}} + 2M_\xi e^{-\Delta\lambda_n}\right) \Delta^{-1} e^{-\rho\Delta}. \end{aligned}$$

Since $\lambda_{N_1} > 0$, $\max_{\Delta \ge \underline{\Delta}} \Delta e^{-\Delta\lambda_{N_1}} < \infty$, whence we can take

$$\zeta_y(n, N_1, y_{N_1+1}) := M_\phi \, |y_{N_1+1}| \left(\max_{\Delta \ge \underline{\Delta}} \Delta e^{-\Delta\lambda_{N_1}} + 2M_\xi e^{-\Delta\lambda_n}\right)$$

in (32). Similarly, we have

$$|\mathcal{K}_\lambda(n)| \quad = \frac{|y_{N_1+1}|}{\Delta |y_n|} \frac{|\phi_{N_1+1} - \phi_n|}{\phi_n} L^2_{\Phi,n}(\phi_{N_1+1}).$$

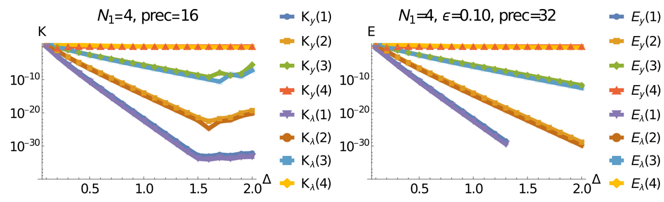
By Assumptions 1-3, $\frac{|y_{N_1+1}|}{|y_n|} \le M_y$, whereas $\frac{|\phi_{N_1+1}-\phi_n|}{\phi_n} \le 1$. Hence, from (26), we again have

$$|\mathcal{K}_\lambda(n)| \le M_y M_\phi \Delta^{-1} e^{-2\Delta\sigma(n,N_1)} =: \zeta_\lambda \Delta^{-1} e^{-\rho\Delta},$$

which concludes the proof. ■

Remark 5: Theorem 2 guarantees exponential decay of $\mathcal{K}_y(n)$ and $\mathcal{K}_\lambda(n)$ provided that $N_1 > N_1^*(\rho)$. This condition is required only for $\mathcal{K}_y(n)$ and follows from the fact that the independent estimates of $L^2_{\Phi,n}(\phi_{N_1+1})$ and $|L'_{\Phi,n}(\phi_n)| = \xi_4$ in (26) and (29), respectively, are combined in bounding $\mathcal{K}_y(n)$, which leads to conservatism. The numerical simulations in the next section show that the predicted exponential decay rate is obtained without choosing N_1 according to (33).

Remark 6: Note that according to Theorem 2, a larger value of the constant $N_1 > N_1^*(\rho)$ actually allows one to obtain a larger constant ρ, thereby leading to faster decay of the condition numbers as $\Delta \to \infty$.



(a) (b)

Fig. 1: (1a) $\mathcal{K}_\lambda, \mathcal{K}_y$ for $N_1 = 4$ and $\lambda_n = n^2$. The asymptotics break down for large Δ, due to inversion of badly conditioned matrices in finite precision computations (16 decimal digits). (1b) ESPRIT algorithm conditioning (see (34)), applied to the sequence $\{y(\Delta k)\}_{k=0}^{2N_1-1}$ with $N_1 = 4$ and $\lambda_n = n^2$. Here we used 32 decimal digits of precision.

V. Numerical Simulations

We provide numerical examples to validate our theory. All simulations are implemented in Wolfram Mathematica [22].

A. Condition numbers

Consider the model (13) with $N_1 = 4$, $N_2 = 1$, $\lambda_n = n^2$, $y_n = 1$ for all n and $t_k = k\Delta$. We use (22) to compute the condition numbers $\mathcal{K}_\lambda(n)$ and $\mathcal{K}_y(n)$, $n \in [N_1]$. The results in Fig. 1a show exponential decay of the condition numbers, with the lowest order parameters ($n = 1$) being the most stable, while the highest order ones ($n = N_1$) are the least stable. Differently from Theorem 2, here we fix N_1 a priori, and still obtain an exponential decay of the condition numbers.

B. ESPRIT algorithm

The ESPRIT algorithm [17] is one of the best-performing methods for exponential fitting. It requires at least $2N_1$ equispaced samples of the signal $y(t)$ of the form (13), and produces estimates of the parameters $\{\lambda_n, y_n\}_{n=1}^{N_1}$. It is known to provide exact solutions in the noiseless case (i.e. $\varepsilon = 0$), and performs close to optimal in the presence of noise, in the context of the so-called super-resolution problem in applied harmonic analysis [16].

We apply ESPRIT to the sequence $\{y(\Delta k)\}_{k=0}^{2N_1-1}$, with the same setup as in Section V-A. In Fig. 1b, we see that the conditioning of the ESPRIT algorithm is consistent with Theorem 2 and the computed condition numbers in Section V-A. We plot the rescaled errors (recall (16)),

$$\mathcal{E}_\lambda(n) = \frac{|\widetilde{\lambda}_n - \lambda_n|}{\varepsilon}, \qquad \mathcal{E}_y(n) = \frac{|\widetilde{y}_n - y_n|}{\varepsilon}, \tag{34}$$

where $\widetilde{\lambda}_n$ and $\widetilde{y}_n$ are the parameter values recovered by ESPRIT, and, furthermore, the $\widetilde{\lambda}_n$'s have been index-matched to the true λ_n's. Here $\varepsilon = 10^{-1}$, and the results were computed with 32 decimal digits of precision.

C. PDE parameter identification

We test the complete procedure on a PDE identification problem. We consider the PDE (1) with constant $p \equiv q \equiv 0.1$. The eigenvalues and eigenfunctions are explicitly given by $\lambda_n = n^2\pi^2 - q$ and $\psi_n(x) = \sqrt{2}\sin(n\pi x)$. The initial condition

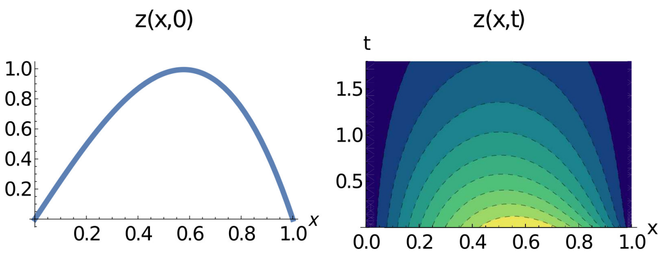


Fig. 2: Numerical solution of the PDE (right) with the specified initial condition (left).

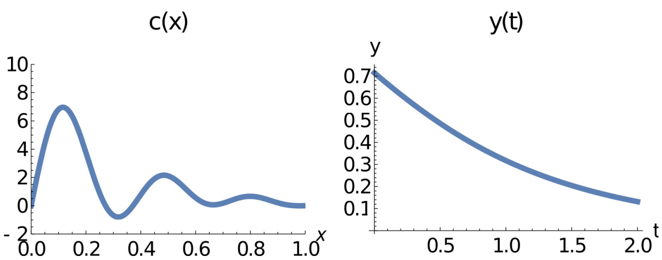
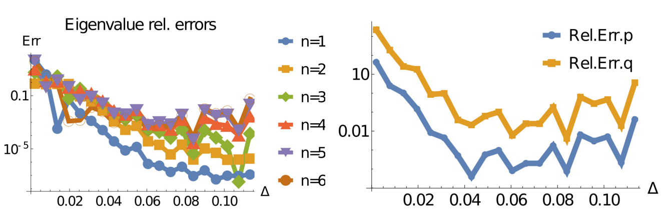


Fig. 3: The measurement filter $c(x)$, $x \in [0,1]$ and the corresponding non-local measurement data $y(t)$, $t \in [0,2]$.

is set to be $z_n(0) = (-1)^{n+1}\left(\sqrt{2}n^3\right)^{-1}$. To solve the PDE, we use the method of lines for space discretization with $N_x = 60$ collocation points and 4th order finite difference approximation, and the resulting ODE system is integrated for $t \in [0,2]$. The resulting solution and the initial condition are plotted in Fig. 2. Our implementation utilized the `NDSolve` library function.

Next, let $c(x) = \sum_{n=1}^{N_1} c_n \psi_n(x) + \varepsilon \sum_{n=N_1+1}^{N_1+2} c_n \psi_n(x)$ where $\{c_n\}_{n=1}^{N_1+2}$ with $c_n \in [1,2]$ are randomly chosen, and $\varepsilon = 10^{-4}$. The measurements $y(t)$ in (2) are computed using global adaptive quadrature as implemented in `NIntegrate` library function. Finally, $y(t)$ is sampled at 1025 equispaced points in $[0,2]$, thus giving a minimal value of $\Delta_{\min} := \frac{1}{512}$. The filter and the sampled measurements are shown in Fig. 3.



(a) (b)

Fig. 4: ESPRIT recovery errors. (4a) Recovery of PDE eigenvalues by ESPRIT, increasing Δ. The relative errors in λ_n for $n = 1, \ldots, N_1$ are plotted. (4b) Recovery errors of p, q, estimated from $\{\widehat{\lambda}_n\}_{n=1}^{N_1}$ by a linear least squares fit.

We apply the ESPRIT algorithm on $\{y(k\Delta)\}_{k=0}^{2N_1-1}$ with varying Δ. The relative errors in the recovered eigenvalues are plotted in Fig. 4a. The deterioration of the error when Δ passes a certain threshold is consistent with our earlier observations due to the finite precision in the computations. Here, all computations are done with 100 decimal digits of precision. Finally, we estimate p, q from the recovered eigenvalues, using the relationship $\lambda_n = \pi^2 n^2 p - q$ by applying linear least squares regression to $\{\widehat{\lambda}_n\}_{n=1}^{N_1}$. The errors in the estimated parameters are plotted in Fig. 4b.

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