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Uniform upper bounds for the cyclicity of the zero solution of the Abel differential equation

Dmitry Batenkov [a,1], Gal Binyamini [b,*,2]

[a] Department of Computer Science, Technion – Israel Institute of Technology, Haifa 32000, Israel
[b] Department of Mathematics, University of Toronto, Toronto, Canada

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Abstract

Given two polynomials P, q we consider the following question: "how large can the index of the first non-zero moment $\widetilde{m}_k = \int_a^b P^k q$ be, assuming the sequence is not identically zero?" The answer K to this question is known as the moment Bautin index, and we provide the first general upper bound: $K \leqslant 2 + \deg q + 3(\deg P - 1)^2$. The proof is based on qualitative analysis of linear ODEs, applied to Cauchy-type integrals of certain algebraic functions.

The moment Bautin index plays an important role in the study of bifurcations of periodic solution in the polynomial Abel equation $y' = py^2 + \varepsilon q y^3$ for p, q polynomials and $\varepsilon \ll 1$. In particular, our result implies that for p satisfying a well-known generic condition, the number of periodic solutions near the zero solution does not exceed $5 + \deg q + 3\deg^2 p$. This is the first such bound depending solely on the degrees of the Abel equation.

* Corresponding author.
E-mail addresses: batenkov@cs.technion.ac.il (D. Batenkov), galbin@gmail.com (G. Binyamini).
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1. Introduction

1.1. Polynomial moments and their Bautin index

Throughout this paper $P, Q \in \mathbb{C}[z]$ will denote a pair of polynomials, and p, q their respective derivatives. We will denote the degrees of P, Q (resp. p, q) by d_P, d_Q (resp. d_p, d_q). We also fix two points $a, b \in \mathbb{C}$.

Two related types of moment sequences corresponding to this data[3] have been considered in the literature,

$$m_k = m_k(P, Q) := \int_a^b P^k(z)Q(z)p(z)\mathrm{d}z, \qquad k = 0, 1, 2, \ldots \tag{1}$$

$$\widetilde{m}_k = \widetilde{m}_k(P, q) := \int_a^b P^k(z)q(z)\mathrm{d}z, \qquad k = 0, 1, 2, \ldots \tag{2}$$

These two moment sequences appear naturally in the study of perturbations of Abel equation, and particularly in the study of bifurcation of periodic solutions from the zero solution. This connection is explained in detail in Section 1.2. We now proceed to define the moment vanishing index, which plays the central role in the study of bifurcations and is the principal subject of the present paper.

Definition 1. We define the *vanishing index* $N(P, Q, a, b)$ to be the first index k such that $m_k(P, Q) \neq 0$, or ∞ if no such k exists. We define the *moment Bautin index* $N(d_P, d_Q, a, b)$ to be the least $k \in \mathbb{N}$ with the property that $N(P, Q, a, b) \geqslant k$ implies $N(P, Q, a, b) = \infty$ for any P, Q with $\deg P \leqslant d_P$ and $\deg Q \leqslant d_Q$.

We define $\widetilde{N}(P, q, a, b)$ and $\widetilde{N}(d_P, d_q, a, b)$ analogously, with respect to the moments $\widetilde{m}_k$.

Remark 2. The moments $m_k(P, Q)$ are polynomials in the coefficients of P, Q. Let R denote the ring of polynomials in these coefficients and $I_k \subset R$ denote the ideal by $m_0, \ldots, m_k$. Then $N(d_P, d_Q, a, b)$ defined above is the first index for which the chain $\{\sqrt{I_k}\}_{k\in\mathbb{N}}$ stabilizes. In particular, from noetherianity it follows that this index is well-defined (finite). An analogous remark holds for $\widetilde{N}(d_P, d_q, a, b)$.

The moment Bautin index has been studied in various special cases, motivated primarily by its relation to perturbations of the Abel equation (see Section 1.2 for an overview). Bounds have been obtained in various special cases, including the cases $d_P = 2, 3$. We refer the reader to [1,2] and references therein for details. However, to our knowledge no general bound has been established. Our main result is the following general bound for the moment Bautin index.

Theorem 1. *For any $d_P, d_Q \in \mathbb{N}$ we have*

$$N(d_P, d_Q, a, b) \leqslant d_Q + 3(d_P - 1)^2. \tag{3}$$

[3] For standard reasons, the moments (1), (2) do not depend on the choice of the integration path from a to b.

Similarly, for any d_P, d_q *we have*

$$\widetilde{N}(d_P, d_q, a, b) \leqslant 2 + d_q + 3(d_P - 1)^2. \tag{4}$$

In particular, we draw the attention of the reader to the fact that the moment sequences depend linearly on the polynomials Q or q, and accordingly the estimate in Theorem 1 is linear in the degrees d_Q, d_q, making it fairly realistic. The estimate depends quadratically on d_P, which also appears to be relatively realistic provided the moment sequences' nonlinear dependence on P.

Following [1], this estimate immediately implies an analogous upper bound for the number of periodic solutions bifurcating from the zero solution for a large class of Abel equations. A precise statement of this type is given in our second main result, Corollary 4.

1.2. *Perturbations of the Abel equation*

The classical Hilbert's 16th problem asks for bounding the number of limit cycles, i.e. isolated closed trajectories, of the polynomial vector field

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= -y + F(x, y), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= x + G(x, y). \end{aligned} \tag{5}$$

The closely related *Poincaré Center-Focus Problem* asks for explicit conditions on the polynomials F, G in order for the system (5) to have a center. These problems remain widely open, although during the years many partial results have been obtained (see [3] for an exposition).

An alternative context for the study of the problems above is provided by the Abel differential equation,

$$y' = p(x)y^2 + q(x)y^3, \quad x \in [a, b] \subset \mathbb{R}, \tag{6}$$

where p, q can be polynomials, trigonometric polynomials or even analytic functions [4]. A periodic solution in this context corresponds to solution $y(x)$ satisfying $y(a) = y(b)$, and a center corresponds to an Abel equation where every solution with a sufficiently small initial condition is periodic. The Abel equation analogue of the Hilbert 16th problem, known as the Smale–Pugh problem, is to bound the number of periodic solutions of (6) in terms of the degrees of p and q. It is generally believed that some (but not all) of the essential difficulties in the study of (5) can be observed in (6), even when one restricts to the case of polynomial coefficients. On the other hand, the polynomial Abel equation allows for several important technical simplifications, and significant progress has been achieved for the Center-Focus problem in this context using tools from polynomial composition algebra and algebraic geometry [5,6].

The Smale–Pugh problem for the polynomial Abel equation remains open. Its infinitesimal version, first suggested in [2], is as follows:

Problem 1. *How many periodic solutions can a small perturbation*

$$y' = p(x)y^2 + \varepsilon q(x)y^3, \quad x \in [a, b] \tag{7}$$

of the "integrable" equation $y' = p(x)y^2$ *have?*

This is an Abel equation analog of the "Infinitesimal Hilbert 16th problem" for which an explicit bound was obtained in [7]. Following [2], in this paper we focus our attention on the periodic solutions bifurcating from *the zero solution* of (7).

The unperturbed equation ($\varepsilon = 0$) is a center if and only if $\int_a^b p(x)\mathrm{d}x = 0$. Thus we may choose the primitive P such that $P(a) = P(b) = 0$. As in the classical case, the study of the bifurcation of periodic solutions as well as the center conditions for the perturbation (7) begins with the study of the first variation of the Poincaré map.

For technical reasons it is customary to consider the "reverse" map from time $x = b$ to time $x = a$. Namely, let $G(y) : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ denote the germ of the analytic map assigning to each initial condition y_b the value $G(y_b) = \eta(a)$, where η is a solution of (7) satisfying $\eta(b) = y_b$. We may view G as a germ of an analytic function in the coefficients of the polynomials p, q and ε as well. Fixed points of G correspond to periodic solutions, and the identical vanishing of $G(y)$ corresponds to a center. An explicit computation [2, Proposition 4.1] gives the expansion

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} G(y) = -y^3 \int_a^b \frac{q(x)}{1 - yP(x)}\mathrm{d}x = \sum_{k=0}^{\infty} \widetilde{m}_k y^{k+3}. \tag{8}$$

As in the classical study of perturbation of Hamiltonian planar systems, it follows from this variational computation that the number of periodic solutions bifurcating from the zero solution of (7) is bounded by the order of zero of the right hand side, i.e. $\widetilde{N}(P, q, a, b) + 3$, assuming that this number is finite. On the other hand, if the first variation vanishes identically then one must in general consider higher order variations in ε, further complicating the study of bifurcating periodic solutions.

A surprising feature of the Abel equation (7) is that for many polynomials p, the vanishing of the first variation (8) automatically implies the identical vanishing of the Poincaré map. Toward this end we recall the following definition.

Definition 3. (See [1].) The polynomials P, Q are said to satisfy the composition condition (PCC) on $[a, b]$ if there exists a polynomial $W(x)$ with $W(a) = W(b)$, and polynomials $\tilde{P}$, $\tilde{Q}$ such that

$$P(x) = \tilde{P}(W(x)), \qquad Q(x) = \tilde{Q}(W(x)).$$

A polynomial P is called "definite" (w.r.t. a, b), if for any polynomial Q, vanishing of all the moments $\widetilde{m}_k(P, q)$ implies PCC for P, Q.

Definite polynomials are ubiquitous. In the deep works [8,6] all counter-examples have been classified and shown to admit a rigid algebraic structure.

Whenever the polynomials P, Q satisfy the PCC, the corresponding Abel equation (6) automatically admits a center, as can be seen by a simple change of variable argument. We thus see that for a definite polynomial P, the vanishing of all moments $\widetilde{m}_k(P, q)$ implies the *identical* vanishing of the Poincaré map $G(y)$. Therefore, in a sense the bifurcation of periodic solutions in (7) is fully controlled by the first variation (8). More formally, the following holds.

Theorem 2. *(See [1, Theorem 2.3].) Let P be a definite polynomial, and fix the parameters a, b, d_q. Then there exist positive $\epsilon = \epsilon(P, d_q)$ and $\delta = \delta(P, d_q)$ such that for any $\|q\| < \epsilon$ with $\deg q \leqslant d_q$, the number of periodic solutions of (6) with $|y(a)| < \delta$ is at most $\widetilde{N}(d_P, d_q, a, b) + 3$.*

As a corollary of Theorem 1 we therefore have the following first general estimate for the number of limit cycles near the zero solution for an Abel equation (6) with $\|q\|$ small.

Corollary 4. *Under the conditions of Theorem 2, the number of periodic solutions is bounded by $5 + d_q + 3d_p^2$.*

1.3. Overview of the proof

It is shown in Section 2 (following [9]) that the second bound in Theorem 1 follows immediately from the first. Our approach to the proof of the first bound is based on the following two observations:

1. The vanishing index $N(P, Q, a, b)$ is essentially equivalent to the order of the zero at $t = \infty$ of the moment generating function $H(t)$ for the moment sequence $\{m_k\}$. It turns out [9] that $H(t)$ admits an analytic expression as a Cauchy type integral for the algebraic function $Q(P^{-1}(z))$.
2. The Cauchy type integral above satisfies a (non-homogeneous) linear differential equation of Fuchsian type [10].

The problem of estimating $N(P, Q, a, b)$ is thus reduced to the study of the order of zero at $t = \infty$ of solutions of certain Fuchsian differential equations. A detailed analysis of the Fuchsian differential operator involved, and elementary considerations concerning its monodromy, allow us to give an a priori upper bound for this order of zero, thus proving Theorem 1.

Remark 5. We thank the anonymous referee for bringing to our attention the paper [11], which studies the maximal possible multiplicity of a zero of generic abelian integrals. While the particular arguments involved are different, the approach of this paper is also based on studying the maximal vanishing orders of solutions for certain linear ODEs.

1.4. Organization of the paper

In Section 2 we introduce moment generating functions for the two moment sequences $\{m_k\}, \{\widetilde{m}_k\}$ which turn out to be Cauchy-type integrals. In Section 3 we give a slightly generalized version of the result of [10] which states that if a function $g(z)$ satisfies a linear ODE $\mathcal{L}g = 0$ then the corresponding Cauchy-type integral $I(t)$ satisfies a non-homogeneous linear ODE $\mathcal{L}I = R$, where R is a rational function of known degree. Subsequently, in Section 4 we explicitly derive the corresponding non-homogeneous ODE for the moment generating functions. Finally in Section 5 we produce estimates for the order of zero the moment generating function at infinity using qualitative methods of linear ODEs.

2. Polynomial moments and generating functions

Recall the notations of Section 1.1. We introduce moment generating functions with the corresponding integral expression for the sequences $\{m_k\}, \{\widetilde{m}_k\}$ as follows:

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$$H(t) = \sum_{k=0}^{\infty} m_k t^{-(k+1)} \qquad H(t) = \int_a^b \frac{Q(z)p(z)}{t - P(z)} \mathrm{d}z, \tag{9}$$

$$\widetilde{H}(t) = \sum_{k=0}^{\infty} \widetilde{m}_k t^{-(k+1)} \qquad \widetilde{H}(t) = \int_a^b \frac{q(z)}{t - P(z)} \mathrm{d}z. \tag{10}$$

Clearly,

$$\operatorname{ord}_\infty H(t) = N(P, Q, a, b) + 1 \qquad \operatorname{ord}_\infty \widetilde{H}(t) = \widetilde{N}(P, q, a, b) + 1. \tag{11}$$

In particular, we have the following.

Proposition 6. *We have*

$$N(d_P, d_Q, a, b) = \sup_{H(t) \not\equiv 0} \operatorname{ord}_{t=\infty} H(t), \tag{12}$$

where the supremum is taken over all pairs P, Q with respective degrees bounded by d_P, d_Q and $H(t)$ denotes the corresponding moment generating function.

It turns out that $H(t)$ and $\widetilde{H}(t)$ are related by a simple formula, which implies in particular that the study of their orders of vanishing at $t = \infty$ are essentially the same [9, Claim, p. 40]. We repeat the argument of [9] in order to obtain an explicit description of relation between these orders.

Lemma 7. *The condition $\widetilde{m}_0 = 0$ is equivalent to $Q(a) = Q(b)$. Moreover, under this condition we have $\widetilde{m}_{k+1} = -(k+1)m_k$ for $k \in \mathbb{N}$. In particular, we have*

$$\widetilde{N}(P, q, a, b) \leqslant N(P, Q, a, b) + 1. \tag{13}$$

Proof. Derivating under the integral sign we have

$$\begin{aligned} \frac{\mathrm{d}H(t)}{\mathrm{d}t} &= -\int_a^b \frac{Q(z)p(z)}{(t - P(z))^2} \mathrm{d}z = -\int_a^b Q \mathrm{d}\left(\frac{1}{t - P(z)}\right) \\ &= -\Big[\frac{Q(z)}{t - P(z)}\Big]_a^b + \int_a^b \frac{q(z)}{t - P(z)} \mathrm{d}z = \frac{Q(a)}{t - P(a)} - \frac{Q(b)}{t - P(b)} + \widetilde{H}(t) \end{aligned} \tag{14}$$

Comparing the t^{-1} coefficient we see that $\widetilde{m}_0 = 0$ if and only if $Q(a) = Q(b)$, and under this condition $\widetilde{m}_{k+1} = -(k+1)m_k$ as claimed. □

The moment generating function (9) has the form of a Cauchy integral. Indeed, choose the curve of integration γ' from a to b in (9) to be some smooth curve avoiding the critical values of

$P(z)$ (except perhaps at the endpoints). Then setting $\gamma = P(\gamma')$ and substituting $w = P(z)$ in (9) we obtain

$$H(t) = \int_\gamma \frac{Q(P^{-1}(w))}{t-w} \mathrm{d}w \tag{15}$$

where $P^{-1}(w)$ denotes the branch of P^{-1} lifting γ to γ'.

3. Cauchy-type integrals and linear differential operators

Let $\mathcal{L}$ be a scalar differential operator,

$$\mathcal{L} = c_r(z)\partial^r + \cdots + c_0(z), \qquad c_0, \ldots, c_r \in \mathbb{C}[z]. \tag{16}$$

Let $\gamma \subset \mathbb{C}$ be a smooth curve, and assume that γ does not pass through the singular points of $\mathcal{L}$, except perhaps at its endpoints. Finally let g be a solution of $\mathcal{L}g = 0$ defined on γ, and assume further that g is bounded on γ (including at the possibly singular endpoints). We denote by p_+, p_- the endpoints of γ.

Then we define the Cauchy-type integral

$$I(t) = \int_\gamma \frac{g(z)}{z-t} \mathrm{d}z \tag{17}$$

It is classically known that $I(t)$ is a holomorphic functions defined on $\mathbb{C} \setminus \gamma$, and moreover that the boundary values I^+ and I^- of $I(t)$ on γ from above and below respectively satisfy $I^+ - I^- = g|_\gamma$. Moreover $I(t)$ can be analytically continued along any path avoiding the endpoints of γ.

Kisunko [10] proved the following (under the extra mild assumption that g is holomorphic at the endpoints of γ).

Proposition 8. *We have $\mathcal{L}I(t) = R(t)$ where $R(t)$ is a rational function having poles of order at most r at p_+, p_- and no other poles on $\mathbb{C}$.*

Sketch of proof. By the classical properties of $I(t)$ mentioned above, $\mathcal{L}I(t)$ is a (possibly multivalued) analytic function on $\mathbb{C} \setminus \{p_+, p_-\}$ with ramifications p_+, p_-, and the difference between the two branches near the branch cut at γ is g. But since $\mathcal{L}g = 0$, the boundary values of $\mathcal{L}I^+$ and $\mathcal{L}I^-$ agree, so $\mathcal{L}I$ is in fact a univalued holomorphic function defined on $\mathbb{C} \setminus \{p_+, p_-\}$. We will show that it has poles of order at most r at p_+, p_- and at most a pole at ∞.

Since $g|_\gamma$ is bounded, we may derive under the integral sign and write

$$\mathcal{L}I(t) = \sum_{k=0}^{r} \frac{(-1)^k}{k!} c_k(t) \int_\gamma \frac{g(z)}{(z-t)^{k+1}} \mathrm{d}z \tag{18}$$

We now show that $\mathcal{L}I(t)$ admits polynomial growth of order at most r at p_+ (and the same arguments work for p_-). It is enough to consider each of the integrals in (18) separately. Moreover, we may assume that γ is a small piece of a smooth curve near p_+ (because the integral over the rest of γ is analytic at p_+). Choose a coordinate system where $p_+ = 0$.

Let M denote an upper bound for $|g(z)|$ on γ. Let t be a point in a punctured neighborhood of p_+. Since $g(z)$ admits analytic continuation along any curve in the punctured neighborhood, may deform γ without changing $\mathcal{L}I(t)$ so that for some positive constants C, D independent of t:

1. For every $z \in \gamma$, we have $|z - t| \geqslant C|t|$ and also $|z - t| > C|z|$.
2. On γ we have $|\mathrm{d}z| \leqslant D\mathrm{d}|z|$.
3. Write $g = \gamma_1 + \gamma_2$ where γ_1 is the part of γ which lies in $\{z : |z| < 2t\}$ and γ_2 is the rest. Then the length of γ_1 is at most $D|t|$, and the length of γ_2 is at most D.

We now estimate

$$
\begin{aligned}
|\int_\gamma \frac{g(z)}{(z-t)^{k+1}}\mathrm{d}z| &\leqslant \int_{\gamma_1} M(C|t|)^{-k-1}\,|\mathrm{d}z| + \int_{\gamma_2} \frac{MD}{(C|z|)^{k+1}}\mathrm{d}|z| \\
&\leqslant \mathrm{length}(\gamma_1) M(C|t|)^{-k-1} + \left[-\frac{MDC^{k+1}}{k}|z|^{-k}\right]_{2|t|}^{\cdots} \leqslant O(|t|^{-k})
\end{aligned}
\tag{19}
$$

proving the claim.

Finally, it is easy to see that $I(t)$ and its derivatives have a zero at $t = \infty$, and since the coefficients of $\mathcal{L}$ are polynomial it follows that $I(t)$ has at most a pole at ∞ as well. □

4. A differential operator for $Q(P^{-1})$

Let V denote the linear space spanned by the d_P branches of the algebraic function $g(z) := Q(P^{-1}(z))$. We denote $r := \dim V$, and note that r may be strictly smaller than d_P. Denote by $p_1, \dots, p_s$ the critical values of P.

4.1. *The operator $\mathcal{L}$*

By a theorem of Riemann [3, Theorem 19.7], there exists a linear r-th order differential operator $\mathcal{L}$, with polynomial coefficients,

$$
\mathcal{L} = c_r(z)\partial^r + \cdots + c_0(z), \qquad c_0, \dots, c_r \in \mathbb{C}[z] \tag{20}
$$

whose solution space coincides with V. Moreover, $\mathcal{L}$ is uniquely determined by the requirement that $c_r, \dots, c_0$ do not share a non-trivial common factor. We recall the construction of $\mathcal{L}$.

Recall that the Wronskian $W(f_1, \dots, f_n)$ of a tuple of functions is defined to be

$$
W(f_1, \dots, f_n) := \det \begin{pmatrix} f_1 & \cdots & f_n \\ \partial f_1 & \cdots & \partial f_n \\ & \vdots & \\ \partial^{n-1} f_1 & \cdots & \partial^{n-1} f_n \end{pmatrix} \tag{21}
$$

Now let $g_1, \ldots, g_r$ denote r branches of $g(z)$ which span V. Then clearly for any $f \in V$ we have $W(g_1, \ldots, g_r, f) = 0$. We define the operator $\widetilde{L}$ given by

$$\widetilde{\mathcal{L}}(f) = \frac{W(g_1, \ldots, g_r, f)}{W_r} = \left[\partial^r + \sum_{k=0}^{r-1} \widetilde{c}_k(z)\partial^k\right] f \qquad \text{where } \widetilde{c}_k = \frac{W_k(g_1, \ldots, g_r)}{W_r(g_1, \ldots, g_r)} \tag{22}$$

where W_i are the minors obtained when expanding the Wronskian $W(g_1, \ldots, g_r, f)$ along the last column. If the monodromy of g along a closed curve γ induces the linear automorphism $M_\gamma : V \to V$ then the corresponding monodromy along γ of each W_k is given by multiplication by $\det M_\gamma$. In particular, the coefficients $\widetilde{c}_k$ are univalued functions.

4.2. *The divisors* $[W_k]$

Let $k = 0, \ldots, r$ and $z_0 \in \mathbb{C}P$. Choose any local representative of the functions $g_1, \ldots, g_r$. Since these functions have at most a finite ramification and moderate growth at z_0, we may expand

$$W_k = \sum_{j=-N}^{\infty} a_{k,j}(z - z_0)^{j/q} \tag{23}$$

where q and N are some natural numbers. Suppose that a_{k,j_0} is the first non-zero coefficient among the $a_{k,j}$. Then we say that the *fractional order of* W_k *at* z_0 is $\operatorname{ord}_{z_0} W_k := j_0/q$. This notion is well-defined: indeed, the monodromy of W_k along any curve is given by multiplication by a non-zero constant and hence does not change the order. We define the fractional divisor $[W_k]$ of W to be

$$[W_k] := \sum_{z \in \mathbb{C}P} \operatorname{ord}_z W_k(z)[z]. \tag{24}$$

This sum is locally-finite, and hence finite. Moreover it is clear that $[\widetilde{c}_k] = [W_k] - [W_r]$. In particular, $\widetilde{c}_k$ admits finitely many singularities of finite order. Since we have already seen that $\widetilde{c}_k$ is univalued, it is in fact a rational function.

We can also write the divisor $[W_k]$ in terms of residues. Indeed, since the monodromy of W_k along any curve is given by multiplication by a constant, the one-form $\mathrm{d}\operatorname{Log} W_k$ is a univalued one-form. It is easy to verify in local coordinates that it in fact has only finitely many poles, all of first order, and

$$[W_k] = \sum_{z \in \mathbb{C}P} \operatorname*{Res}_z(\mathrm{d}\operatorname{Log} W_k)[z]. \tag{25}$$

For any divisor $D = \sum n_i[z_i]$ we denote $D_{z_i} = n_i$ and

$$\deg D = \sum n_i, \qquad D^+ = \sum_{n_i \geqslant 0} n_i[z_i], \quad D^- = -\sum_{n_i \leqslant 0} n_i[z_i]. \tag{26}$$

In particular, it follows from the above that $\deg[W_k] = 0$.

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4.3. *An estimate for* $\deg[W_r]^+$

Our next goal is to estimate $\deg[W_r]^+$. Since $[W_r]$ is principal, it will suffice to estimate $\deg[W_r]^-$. Recall that $W_r = W(g_1, \dots, g_r)$ where $g_k = Q(P_k^{-1}(z))$ and $P_1^{-1}(z), \dots, P_r^{-1}(z)$ denote r different branches of $P^{-1}(z)$. If $z \in \mathbb{C}$ is not a critical value of P then these functions are all holomorphic around z, and hence $[W_r]_z$ is non-negative.

Let the critical value p_i have exactly $m_i < d_P$ preimages, and write $b_i := d_P - m_i$ for the number of critical points (counted with multiplicities) over p_i. Then at most $2b_i$ of the branches g_k may be ramified at p_i. We expand the determinant defining W_r and note that:

- since g_k is bounded, its order is non-negative;
- differentiation can decrease the order by at most 1;
- differentiation cannot decrease the order below zero for holomorphic g_k.

We thus conclude that

$$\operatorname{ord}_{p_i} W_r > (-r+1) + \cdots + (-r+\nu), \qquad \text{where } \nu = \min(r, 2b_i).$$

Since $b_1 + \cdots + b_s = d_P - 1$, it is not hard to see that the maximal value for the following sum is obtained when $b_i = 1$ for $i = 1, \dots, s$, and in any case

$$\sum_{i=1}^{s} \operatorname{ord}_{p_i} W_r > -(2d_P - 3)(d_P - 1). \tag{27}$$

It remains to estimate the order of W_r at ∞. Choose a coordinate w around ∞ such that $P(w) = w^{-d_P}$. Then any branch of $Q(P^{-1}(w))$ has the Puiseux expansion

$$Q(P^{-1}(w)) = Q(w^{-1/d_P}) = \alpha w^{-d_Q/d_P} + \cdots, \qquad \alpha \neq 0 \tag{28}$$

where $\cdots$ denote higher order terms. Moreover, the derivative $\partial_z = -w^2 \partial_w$ increases the order of zero at $w = 0$ by at least one. Expanding the determinant defining W_r we see that

$$\operatorname{ord}_\infty W_r \geqslant -\frac{r d_Q}{d_P} + \frac{r(r-1)}{2}. \tag{29}$$

In conclusion, we have

$$\deg[W_r]^+ = \deg[W_r]^- \leqslant \frac{d_Q r}{d_P} + (2d_P - 3)(d_P - 1) - \frac{r(r-1)}{2}. \tag{30}$$

4.4. *An estimate for* $\deg c_r$

We wish to derive an estimate for the number of singularities of $\mathcal{L}$, or more specifically for $\deg c_r$. By definition, c_r is a polynomial and $[c_r]^+$ is the least common upper bound for $[\widetilde{c}_0]^-, \dots, [\widetilde{c}_{r-1}]^-$ in (22). Recall that $[\widetilde{c}_k] = [W_k] - [W_r]$.

We first note that $\mathcal{L}$ is a Fuchsian operator. Indeed, since the solutions of $\mathcal{L}$, being algebraic functions, have moderate growth at each singularity, this follows from a theorem of Fuchs

[3, Theorem 19.20]. Thus by definition the order of $[c_r]$ at any point $p \in \mathbb{C}$ cannot exceed r. We will apply this to the points $p_1, \dots, p_s$.

Let now $z \in \mathbb{C}$ and $z \notin \{p_1, \dots, p_s\}$. Then the branches $g_1, \dots, g_r$ are holomorphic at z, so z is not a point of $[W_0]^-, \dots, [W_{r-1}]^-$. In other words, z can only be a point of $[\widetilde{c}_0]^-, \dots, [\widetilde{c}_{r-1}]^-$ if it comes from $[W_r]^+$. Thus we see that

$$[c_r]^+ \leqslant \sum_{i=1}^{s} r[p_i] + [W_r]^+. \tag{31}$$

Using (30) and noting that $s \leqslant d_P - 1$ and $r \leqslant d_P$, we have the following proposition.

Proposition 9. *The following estimate holds,*

$$\deg c_r = \deg[c_r]^+ \leqslant \frac{d_Q r}{d_P} + 3(d_P - 1)^2 - \frac{r(r-1)}{2}. \tag{32}$$

5. Estimate for the order of $H(t)$ at infinity

Recall from Section 2 that $H(t)$ denotes the moment generating function (9), which can be represented (around $t = \infty$) as a Cauchy-type integral (15) of the algebraic function $g(z) = Q(P^{-1}(z))$. Recall from Section 4 that V denotes the linear span of the branches of $g(z)$ with $r := \dim V$ and $\mathcal{L}$ the differential operator (20) satisfying $V = \ker \mathcal{L}$.

Proposition 10. *If $H(t) \not\equiv 0$ then $\mathcal{L}H(t) \not\equiv 0$.*

Proof. Assume that $\mathcal{L}H(t) \equiv 0$. Then $H(t) \in V$. Moreover, $H(t)$ is holomorphic at $t = \infty$, and in particular it is invariant under the monodromy around infinity M_∞ and hence also under the operator

$$T_\infty : V \to V, \qquad T_\infty := \frac{1}{d_P} \sum_{k=0}^{d_P - 1} M_\infty^k. \tag{33}$$

Recall that $g(z) = Q(P^{-1}(z))$ and $P^{-1}(z)$ has cyclic monodromy at ∞. It follows that the image of T_∞ is one-dimensional and spanned by

$$\operatorname{Im} T_\infty = \mathbb{C}\{S\}, \qquad S(t) := \sum_{w : P(w) = t} Q(w). \tag{34}$$

Moreover, $S(t)$ is a polynomial: for instance, it is has no poles on $\mathbb{C}$ and moderate growth at ∞. We conclude that $H(t)$ is a polynomial. Finally, $H(t)$ has a zero at $t = \infty$ by definition, and since it is also a polynomial it follows that $H(t) \equiv 0$, contradicting the hypothesis. □

Let $D = t\partial_t$ denote the Euler operator, and recall that it also gives the Euler operator at $t = \infty$ (up to a sign). The following proposition describes the behavior of $\mathcal{L}$ around infinity.

Proposition 11. *We may write*

$$\mathcal{L}(t) = u(t)\widehat{\mathcal{L}} \qquad \widehat{\mathcal{L}} := D^r + \widehat{c}_{r-1}D^{r-1} + \cdots + \widehat{c}_0, \tag{35}$$

where $\widehat{c}_{r-1}, \dots, \widehat{c}_0$ are rational functions, holomorphic at $t = \infty$, and $u(t)$ is a rational function satisfying

$$\operatorname{ord}_\infty u \geqslant -\left[\frac{d_Q r}{d_P} + 3(d_P - 1)^2 - \frac{r(r+1)}{2}\right]. \tag{36}$$

Proof. The existence of an expression (35) is a direct consequence of the fact that $\mathcal{L}$ is a Fuchsian operator at $t = \infty$ (see [3, Proposition 19.18]). Using Proposition 9 we have

$$\operatorname{ord}_\infty u = r - \deg c_r \geqslant -\left[\frac{d_Q r}{d_P} + 3(d_P - 1)^2 - \frac{r(r+1)}{2}\right], \tag{37}$$

as claimed. □

Finally we have the following estimate.

Lemma 12. *If $H(t) \not\equiv 0$ then*

$$\operatorname{ord}_\infty H(t) \leqslant \frac{d_Q r}{d_P} + 3(d_P - 1)^2 - \frac{r(r-3)}{2}. \tag{38}$$

Proof. Using Proposition 8 we have $\mathcal{L}H(t) = R(t)$, where $R(t)$ has at most two poles of order r in $\mathbb{C}$. Moreover, by Proposition 10 $R(t)$ is non-zero. It follows that $\operatorname{ord}_\infty R(t) \leqslant 2r$. Using Proposition 11 we have

$$\begin{aligned}
\operatorname{ord}_\infty(\widehat{\mathcal{L}}H(t)) &= \operatorname{ord}_\infty R(t) - \operatorname{ord}_\infty u(t) \\
&\leqslant 2r + \frac{d_Q r}{d_P} + 3(d_P - 1)^2 - \frac{r(r+1)}{2} \\
&\leqslant \frac{d_Q r}{d_P} + 3(d_P - 1)^2 - \frac{r(r-3)}{2}.
\end{aligned}$$

It remains only to note that the application of $\widehat{\mathcal{L}}$ cannot decrease the order of zero, and the claim follows. □

Finally we complete the proof of our main result.

Proof of Theorem 1. If $H(t) \not\equiv 0$ then by Lemma 12

$$\operatorname{ord}_\infty H(t) \leqslant d_Q + 3(d_P - 1)^2, \tag{39}$$

and the claim for $N(d_P, d_Q, a, b)$ follows by Proposition 6. The claim for $\widetilde{N}(d_P, d_q, a, b)$ then follows from Lemma 7, noting that $d_Q = d_q + 1$. □

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