

# Local and global geometry of Prony systems and Fourier reconstruction of piecewise-smooth functions

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**Abstract** Many reconstruction problems in signal processing require solution of a certain kind of nonlinear systems of algebraic equations, which we call Prony systems. We study these systems from a general perspective, addressing questions of global solvability and stable inversion. Of special interest are the so-called “near-singular” situations, such as a collision of two closely spaced nodes.

We also discuss the problem of reconstructing piecewise-smooth functions from their Fourier coefficients, which is easily reduced by a well-known method of K.Eckhoff to solving a particular Prony system. As we show in the paper, it turns out that a modification of this highly nonlinear method can reconstruct the jump locations and magnitudes of such functions, as well as the pointwise values between the jumps, with the maximal possible accuracy.

## 1 Introduction

In many applications, it is often required to reconstruct an unknown signal from a small number of measurements, utilizing some a-priori knowledge about the signal structure. Such problems arose (and continue to arise) in recent years under several names in different fields, such as Finite Rate of Innovation, super-resolution, sub-Nyquist sampling and Algebraic Signal Reconstruction [6, 8, 9, 13, 14, 15, 19, 23, 31, 34]. One underlying connection between these problems is that almost all of them require solution of a certain kind of nonlinear systems of algebraic equations, which we call Prony systems. We therefore consider the study of this system to be an important topic. In particular, questions of solvability, uniqueness, including in near-singular situations, as well as stability of reconstruction in the presence of noise turn out to be non-trivial and requiring a delicate study of some related algebraic-geometric structures.

This paper consists of two parts. In the first part, we consider the general Prony system. First, we present a necessary and sufficient condition for the system to be solvable. Next,

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we give simple estimate of the stability of inversion in a “regular” setting. Finally, we consider inversion in several “near-singular” situations, and in particular the practically important situation of colliding nodes. We show that a reparametrization in the basis of divided finite differences turns the problem into a well-posed one in this setting.

In the second part of the paper, we present our recent solution to a conjecture posed by K.Eckhoff in 1995 [17], which asks for an algorithm to reconstruct a piecewise-smooth function with unknown discontinuity locations from its first Fourier coefficients. While the problem of defeating the Gibbs phenomenon received much attention in the last decades (see [1, 2, 11, 17, 18, 21, 22, 24, 25, 33, 35] and references therein), the question of attaining maximal possible accuracy of reconstruction remained open. We show how the Algebraic Reconstruction approach, and in particular an accurate solution of a certain Prony system, provides the required approximation rate.

## 2 The Prony problem

Prony system appears as we try to solve a very simple “algebraic signal reconstruction” problem of the following form: assume that the signal  $F(x)$  is known to be a linear combination of shifted  $\delta$ -functions:

$$F(x) = \sum_{j=1}^d a_j \delta(x - x_j). \quad (1)$$

We shall use as measurements the polynomial moments:

$$m_k = m_k(F) = \int x^k F(x) dx. \quad (2)$$

After substituting  $F$  into the integral defining  $m_k$  we get

$$m_k(F) = \int x^k \sum_{j=1}^d a_j \delta(x - x_j) dx = \sum_{j=1}^d a_j x_j^k.$$

Considering  $a_j$  and  $x_j$  as unknowns, we obtain equations

$$m_k(F) = \sum_{j=1}^d a_j x_j^k, \quad k = 0, 1, \dots \quad (3)$$

This infinite set of equations (or its part, for  $k = 0, 1, \dots, 2d - 1$ ), is called Prony system. It can be traced at least to R. de Prony (1795, [30]) and it is used in a wide variety of theoretical and applied fields. See [3] for an extensive bibliography on the Prony method.

In writing Prony system (3) we have assumed that all the nodes  $x_1, \dots, x_d$  are pairwise different. However, as a right-hand side  $\mu = (m_0, \dots, m_{2d-1})$  of (3) is provided by the actual measurements of the signal  $F$ , we cannot guarantee a priori, that this condition is satisfied for the solution. Moreover, we shall see below that multiple nodes may naturally appear in the solution process. In order to incorporate possible collisions of the nodes, we consider “confluent Prony systems”.

Assume that the signal  $F(x)$  is a linear combination of shifted  $\delta$ -functions and their derivatives:

$$F(x) = \sum_{j=1}^s \sum_{\ell=0}^{d_j-1} a_{j,\ell} \delta^{(\ell)}(x - x_j). \quad (4)$$

**Definition 1** For  $F(x)$  as above, the vector  $D(F) = (d_1, \dots, d_s)$  is the *multiplicity vector* of  $F$ ,  $s = s(F)$  is its *degree* and  $d = \sum_{j=1}^s d_j$  is its *order*. For avoiding ambiguity in these definitions, it is always understood that  $a_{j,d_j-1} \neq 0$  for all  $j = 1, \dots, s$ .

For the moments  $m_k = m_k(F) = \int x^k F(x) dx$  we now get

$$m_k = \sum_{j=1}^s \sum_{\ell=0}^{d_j-1} a_{j,\ell} \frac{k!}{(k-\ell)!} x_j^{k-\ell}.$$

Considering  $x_i$  and  $a_{j,\ell}$  as unknowns, we obtain a system of equations

$$\sum_{j=1}^s \sum_{\ell=0}^{d_j-1} \frac{k!}{(k-\ell)!} a_{j,\ell} x_j^{k-\ell} = m_k, \quad k = 0, 1, \dots, 2d-1, \quad (5)$$

which is called a confluent Prony system of order  $d$  with the multiplicity vector  $D = (d_1, \dots, d_s)$ . The original Prony system (3) is a special case of the confluent one, with  $D$  being the vector  $(1, \dots, 1)$  of the length  $d$ .

The system (5) arises also in the problem of reconstructing a planar polygon  $P$  (or even an arbitrary semi-analytic *quadrature domain*) from its moments

$$m_k(\chi_P) = \iint_{\mathbb{R}^2} z^k \chi_P dx dy, \quad z = x + iy,$$

where  $\chi_P$  is the characteristic function of the domain  $P \subset \mathbb{R}^2$ . This problem is important in many areas of science and engineering [23]. The above yields the confluent Prony system

$$m_k = \sum_{j=1}^s \sum_{i=0}^{d_j-1} c_{i,j} k(k-1) \cdots (k-i+1) z_j^{k-i}, \quad c_{i,j} \in \mathbb{C}, \quad z_j \in \mathbb{C} \setminus \{0\}.$$

As we shall see below, if we start with the measurements  $\mu(F) = \mu = (m_0, \dots, m_{2d-1})$ , then a natural setting of the problem of solving the Prony system is the following:

**Problem 1 (Prony problem of order  $d$ )** *Given the measurements*

$$\mu = (m_0, \dots, m_{2d-1}) \in \mathbb{C}^{2d}$$

*in the right hand side of (5), find the multiplicity vector  $D = (d_1, \dots, d_s)$  of order  $r = \sum_{j=1}^s d_j \leq d$ , and find the unknowns  $x_j$  and  $a_{j,\ell}$ , which solve the corresponding confluent Prony system (5) with the multiplicity vector  $D$ .*

It is extremely important in practice to have a *stable method of inversion*. Many research efforts are devoted to this task (see e.g. [4, 12, 15, 28, 29, 32] and references therein). A basic question here is the following.

**Problem 2 (Noisy Prony problem)** Given the *noisy* measurements

$$\tilde{\mu} = (\tilde{m}_0, \dots, \tilde{m}_{2d-1}) \in \mathbb{C}^{2d}$$

and an estimate of the error  $|\tilde{m}_k - m_k| \leq \varepsilon_k$ , solve Problem 1 so as to minimize the reconstruction error.

### 3 Solving the Prony problem

#### 3.1 Prony mapping

Let us introduce some notations which will be useful in subsequent treatment.

**Definition 2** For each  $w = (x_1, \dots, x_d) \in \mathbb{C}^d$ , let  $s = s(w)$  be the number of distinct coordinates  $\tau_j$ ,  $j = 1, \dots, s$ , and denote  $T(w) = (\tau_1, \dots, \tau_s)$ . The multiplicity vector is  $D = D(w) = (d_1, \dots, d_s)$ , where  $d_j$  is the number of times the value  $\tau_j$  appears in  $\{x_1, \dots, x_d\}$ . The order of the values in  $T(w)$  is defined by their order of appearance in  $w$ .

*Example 1* For  $w = (3, 1, 2, 1, 0, 3, 2)$  we have  $s = 4$ ,  $T(w) = (3, 1, 2, 0)$  and  $D(w) = (2, 2, 2, 1)$ .

*Remark 1* Note the slight abuse of notations between Definition 1 and Definition 2. Note also that the *order* of  $D(w)$  equals to  $d$  **for all**  $w \in \mathbb{C}^d$ .

**Definition 3** For each  $w \in \mathbb{C}^d$ , let  $s = s(w)$ ,  $T(w) = (\tau_1, \dots, \tau_s)$  and  $D(w) = (d_1, \dots, d_s)$  be as in Definition 2. We denote by  $V_w$  the vector space of dimension  $d$  containing the linear combinations

$$g = \sum_{j=1}^s \sum_{\ell=0}^{d_j-1} \gamma_{j,\ell} \delta^{(\ell)}(x - \tau_j) \quad (6)$$

of  $\delta$ -functions and their derivatives at the points of  $T(w)$ . The ‘‘standard basis’’ of  $V_w$  is given by the distributions

$$\delta_{j,\ell} = \delta^{(\ell)}(x - \tau_j), \quad j = 1, \dots, s(w); \ell = 0, \dots, d_j - 1. \quad (7)$$

**Definition 4** The Prony space  $\mathcal{P}_d$  is the vector bundle over  $\mathbb{C}^d$ , consisting of all the pairs

$$(w, g) : \quad w \in \mathbb{C}^d, \quad g \in V_w.$$

The topology on  $\mathcal{P}_d$  is induced by the natural embedding  $\mathcal{P}_d \subset \mathbb{C}^d \times \mathcal{D}$ , where  $\mathcal{D}$  is the space of distributions on  $\mathbb{C}$  with its standard topology.

Finally, we define the Prony mapping  $\mathcal{PM}$  which encodes the Prony problem.

**Definition 5** The Prony mapping  $\mathcal{PM} : \mathcal{P}_d \rightarrow \mathbb{C}^{2d}$  for  $(w, g) \in \mathcal{P}_d$  is defined as follows:

$$\mathcal{PM}((w, g)) = (m_0, \dots, m_{2d-1}) \in \mathbb{C}^{2d}, \quad m_k = m_k(g) = \int x^k g(x) dx.$$

Therefore, a formal solution of the Prony problem is given by the inversion of the Prony mapping  $\mathcal{PM}$ .

Finally, let us recall an important type of matrices which play a central role in what follows.

**Definition 6** Let  $(x_1, \dots, x_s) \in \mathbb{C}^s$  and  $D = (d_1, \dots, d_s)$  with  $d = \sum_{j=1}^s d_j$  be given. The  $d \times d$  confluent Vandermonde matrix is

$$V = V(x_1, d_1, \dots, x_s, d_s) = \begin{bmatrix} \mathbf{v}_{1,0} & \mathbf{v}_{2,0} & \dots & \mathbf{v}_{s,0} \\ \mathbf{v}_{1,1} & \mathbf{v}_{2,1} & \dots & \mathbf{v}_{s,1} \\ & & \dots & \\ \mathbf{v}_{1,d-1} & \mathbf{v}_{2,d-1} & \dots & \mathbf{v}_{s,d-1} \end{bmatrix} \quad (8)$$

where the symbol  $\mathbf{v}_{j,k}$  denotes the following  $1 \times d_j$  row vector

$$\mathbf{v}_{j,k} \stackrel{\text{def}}{=} [x_j^k, kx_j^{k-1}, \dots, k(k-1)\dots(k-d_j)x_j^{k-d_j+1}].$$

The matrix  $V$  defines the linear part of the confluent Prony system (5), namely,

$$V(x_1, d_1, \dots, x_s, d_s) \begin{bmatrix} a_{1,0} \\ \vdots \\ a_{1,d_1-1} \\ \vdots \\ a_{s,d_s-1} \end{bmatrix} = \begin{bmatrix} m_0 \\ m_1 \\ \vdots \\ m_{d-1} \end{bmatrix}. \quad (9)$$

### 3.2 Padé problem and the solvability set

It can be shown that the solution to Problem 1 is equivalent to solving the well-known Padé approximation problem. While this connection is extremely important and insightful, we do not provide the details here for the sake of brevity. Let us only mention the following result.

**Proposition 1** *The tuple*

$$\left\{ s, D = (d_1, \dots, d_s), r = \sum_{j=1}^s d_j \leq d, X = \{x_j\}_{j=1}^s, A = \{a_{j,\ell}\}_{j=1, \dots, s; \ell=0, \dots, d_j-1} \right\}$$

*is a solution to Problem 1 with right-hand side*

$$\mu = (m_0, \dots, m_{2d-1}) \in \mathbb{C}^{2d}$$

if and only if  $(m_0, \dots, m_{2d-1})$  are the first  $2d$  Taylor coefficients at  $z = \infty$  of the rational function

$$R_{D,X,A}(z) = \sum_{j=1}^s \sum_{\ell=1}^{d_j} (-1)^{\ell-1} (\ell-1)! \frac{a_{j,\ell}}{(z-x_j)^\ell} = \sum_{k=0}^{2d-1} \frac{m_k}{z^{k+1}} + O(z^{-2d-1}).$$

The function  $R_{D,X,A}(z)$  is the Stieltjes transform of the corresponding signal  $F(x) = \sum_{j=1}^s \sum_{\ell=0}^{d_j-1} a_{j,\ell} \delta^{(\ell)}(x-x_j)$ , i.e.

$$R_{D,X,A}(z) = \int_{-\infty}^{\infty} \frac{F(x) dx}{z-x}.$$

Using this correspondence, it is not difficult to prove the following result (see [10]).

**Theorem 1** *Let the right-hand side  $(m_0, \dots, m_{2d-1})$  of Problem 1 be given. Let  $\tilde{M}_d$  denote the  $d \times (d+1)$  Hankel matrix*

$$\tilde{M}_d = \begin{bmatrix} m_0 & m_1 & m_2 & \dots & m_d \\ m_1 & m_2 & m_3 & \dots & m_{d+1} \\ \dots & \dots & \dots & \dots & \dots \\ m_{d-1} & m_d & m_{d+1} & \dots & m_{2d-1} \end{bmatrix}.$$

For each  $e \leq d$ , denote by  $\tilde{M}_e$  the  $e \times (e+1)$  submatrix of  $\tilde{M}_d$  formed by the first  $e$  rows and  $e+1$  columns, and let  $M_e$  denote the corresponding square matrix.

Let  $r \leq d$  be the rank of  $\tilde{M}_d$ . Then Problem 1 is solvable if and only if the upper left minor  $|M_r|$  of  $\tilde{M}_d$  is non-zero. The solution, if it exists, is unique, up to a permutation of the nodes  $\{x_j\}$ . The multiplicity vector  $D = (d_1, \dots, d_s)$ ,  $\sum_{j=1}^s d_j = r$ , of the resulting confluent Prony system of order  $r$  is the multiplicity vector of the poles of the rational function  $R_{D,X,A}(z)$ , solving the Padé problem in Proposition 1.

As a corollary we get a complete description of the right-hand side data  $\mu \in \mathbb{C}^{2d}$  for which the Prony problem is solvable (unsolvable). Define for  $r = 1, \dots, d$  sets  $\Sigma_r \subset \mathbb{C}^{2d}$  (respectively,  $\Sigma'_r \subset \mathbb{C}^{2d}$ ) consisting of  $\mu \in \mathbb{C}^{2d}$  for which the rank of  $\tilde{M}_d = r$  and  $|M_r| \neq 0$  (respectively,  $|M_r| = 0$ ). The set  $\Sigma_r$  is a difference  $\Sigma_r = \Sigma_r^1 \setminus \Sigma_r^2$  of two algebraic sets:  $\Sigma_r^1$  is defined by vanishing of all the  $s \times s$  minors of  $\tilde{M}_d$ ,  $r < s \leq d$ , while  $\Sigma_r^2$  is defined by vanishing of  $|M_r|$ . In turn,  $\Sigma'_r = \Sigma_r^1 \setminus \Sigma_r'^2$ , with  $\Sigma_r^1 = \Sigma_r^1 \cap \Sigma_r^2$  and  $\Sigma_r'^2$  defined by vanishing of all the  $r \times r$  minors of  $\tilde{M}_d$ . The union  $\Sigma_r \cup \Sigma'_r$  consists of all  $\mu$  for which the rank of  $\tilde{M}_d = r$ , which is  $\Sigma_r^1 \setminus \Sigma_r'^2$ .

**Corollary 1** *The set  $\Sigma$  (respectively,  $\Sigma'$ ) of  $\mu \in \mathbb{C}^{2d}$  for which the Prony problem is solvable (respectively, unsolvable) is the union  $\Sigma = \cup_{r=1}^d \Sigma_r$  (respectively,  $\Sigma' = \cup_{r=1}^d \Sigma'_r$ ). In particular,  $\Sigma' \subset \{\mu \in \mathbb{C}^{2d}, \det M_d = 0\}$ .*

So for a generic right hand side  $\mu$  we have  $|M_d| \neq 0$ , and the Prony problem is solvable. On the algebraic hypersurface of  $\mu$  for which  $|M_d| = 0$ , the Prony problem is solvable if  $M_{d-1} \neq 0$ , etc.

### 3.3 Stable inversion away from singularities

Consider Problem 2 at some interior point  $\mu_0 \in \Sigma$ . By definition,  $\mu_0 \in \Sigma_{r_0}$  for some  $r_0 \leq d$ . Let  $(w_0, g_0) = \mathcal{PM}^{-1}(\mu_0)$ . Assume for a moment that the multiplicity vector  $D_0 = D(g_0) = (d_1, \dots, d_{s_0})$ ,  $\sum_{j=1}^{s_0} d_j = r_0$ , has a non-trivial collision pattern, i.e.  $d_j > 1$  for at least one  $j = 1, \dots, s_0$ . It means, in turn, that the function  $R_{D_0, X, A}(z)$  has a pole of multiplicity  $d_j$ . Evidently, there exists an arbitrarily small perturbation  $\tilde{\mu}$  of  $\mu_0$  for which this multiple pole becomes a cluster of single poles, thereby changing the multiplicity vector to some  $D' \neq D_0$ . While we address this problem in Section 4 via the bases of divided differences, in this section we consider a “restricted” Prony problem.

**Definition 7** Let  $\mathcal{PM}(w_0, g_0) = \mu_0 \in \Sigma_{r_0}$  with  $D(g_0) = D_0$  and  $s(g_0) = s_0$ . Let  $\mathcal{P}_{D_0}$  denote the following subbundle of  $\mathcal{P}_d$  of dimension  $s_0 + r_0$ :

$$\mathcal{P}_{D_0} = \{(w, g) \in \mathcal{P}_d : D(g) = D_0\}.$$

The restricted Prony mapping  $\mathcal{PM}_{D_0}^* : \mathcal{P}_{D_0} \rightarrow \mathbb{C}^{s_0+r_0}$  is the composition

$$\mathcal{PM}_{D_0}^* = \pi \circ \mathcal{PM} \upharpoonright_{\mathcal{P}_{D_0}},$$

where  $\pi : \mathbb{C}^{2d} \rightarrow \mathbb{C}^{s_0+r_0}$  is the projection map on the first  $s_0 + r_0$  coordinates.

Inverting this  $\mathcal{PM}_{D_0}^*$  represents the solution of the confluent Prony system (5) with fixed structure  $D_0$  from the first  $k = 0, 1, \dots, s_0 + r_0 - 1$  measurements.

**Theorem 2 ([12])** Let  $\mu_0^* = \mathcal{PM}_{D_0}^*((w_0, g_0)) \in \mathbb{C}^{s_0+r_0}$  with the unperturbed solution  $g_0 = \sum_{j=1}^{s_0} \sum_{\ell=0}^{d_j-1} a_{j,\ell} \delta^{(\ell)}(x - \tau_j)$ . In a small neighborhood of  $(w_0, g_0) \in \mathcal{P}_{D_0}$ , the map  $\mathcal{PM}_{D_0}^*$  is invertible. Consequently, for small enough  $\varepsilon$ , the restricted Prony problem with input data  $\tilde{\mu}^* \in \mathbb{C}^{r_0+s_0}$  satisfying  $\|\tilde{\mu}^* - \mu_0^*\| \leq \varepsilon$  has a unique solution. The error in this solution satisfies

$$\begin{aligned} |\Delta a_{j,\ell}| &\leq \frac{2}{\ell!} \left(\frac{2}{\delta}\right)^{s_0+r_0} \left(\frac{1}{2} + \frac{s_0+r_0}{\delta}\right)^{d_j-\ell} \left(1 + \frac{|a_{j,\ell-1}|}{|a_{j,d_j-1}|}\right) \varepsilon, \\ |\Delta \tau_j| &\leq \frac{2}{d_j!} \left(\frac{2}{\delta}\right)^{s_0+r_0} \frac{1}{|a_{j,d_j-1}|} \varepsilon, \end{aligned}$$

where  $\delta \stackrel{\text{def}}{=} \min_{i \neq j} |\tau_i - \tau_j|$  (for consistency we take  $a_{j,-1} = 0$  in the above formula).

*Proof (outline)* The Jacobian of  $\mathcal{PM}_{D_0}^*$  can be easily computed, and it turns out to be equal to the product

$$\mathcal{JP}_{D_0}^* = V(\tau_1, d_1 + 1, \dots, \tau_{s_0}, d_{s_0} + 1) \text{diag}\{E_j\}$$

where  $V$  is the confluent Vandermonde matrix (8) on the nodes  $(\tau_1, \dots, \tau_{s_0})$  and multiplicity vector

$$\tilde{D}_0 = (d_1 + 1, \dots, d_{s_0} + 1),$$

while  $E$  is the  $(d_j + 1) \times (d_j + 1)$  block

$$E_j = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & a_{j,0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{j,d_j-1} \end{bmatrix}.$$

Since  $\mu_0 \in \Sigma_r$ , the highest order coefficients  $a_{j,d_j-1}$  are nonzero. Furthermore, since all the  $\tau_j$  are distinct, the matrix  $V$  is nonsingular. Local invertability follows. To estimate the norm of the inverse, use bounds from [7].  $\square$

Let us stress that we are not aware of any general method of inverting  $\mathcal{PM}_{D_0}^*$ , i.e. solving the restricted confluent Prony problem with the smallest possible number of measurements. As we shall see below in Section 5, such a method exists for a very special case of a single point, i.e.  $s = 1$ .

## 4 Prony inversion near singularities

### 4.1 Collision singularities and finite differences

Collision singularities occur in Prony systems as some of the nodes  $x_i$  in the signal  $F(x) = \sum_{i=1}^d a_i \delta(x - x_i)$  approach one another. This happens for  $\mu$  near the discriminant stratum  $\Delta \subset \mathbb{C}^{2d}$  consisting of those  $(m_0, \dots, m_{2d-1})$  for which some of the coordinates  $\{x_j\}$  in the solution collide, i.e. the function  $R_{D,X,A}(z)$  has multiple poles (or, nontrivial multiplicity vector  $D$ ). As we shall see below, typically, as  $\mu$  approaches  $\mu_0 \in \Delta$ , i.e. some of the nodes  $x_i$  collide, the corresponding coefficients  $a_i$  tend to infinity. Notice, that all the moments  $m_k = m_k(F)$  remain bounded. This behavior creates serious difficulties in solving “near-colliding” Prony systems, both in theoretical and practical settings. Especially demanding problems arise in the presence of noise. The problem of improvement of resolution in reconstruction of colliding nodes from noisy measurements appears in a wide range of applications. It is usually called a “super-resolution problem” and a lot of recent publications are devoted to its investigation in various mathematical and applied settings. See [13] and references therein for a very partial sample.

Here we continue our study of collision singularities in Prony systems, started in [36]. The full details will be published in [10]. Our approach uses bases of finite differences in the Prony space  $\mathcal{P}_d$  in order to “resolve” the linear part of collision singularities. In these bases the coefficients do not blow up any more, as some of the nodes collide.

Let  $\mu_0 \in \Sigma_d$ . Consider the noisy Prony problem in a neighborhood of the exact solution  $(w_0, g_0) = \mathcal{PM}^{-1}(\mu_0)$ . As explained in Section 3.3, if  $D(w_0)$  is non-trivial, then there will always be a multiplicity-destroying perturbation, no matter how small a neighborhood. Assume that the node vector  $w = w(\tilde{\mu})$  is determined, and consider the linear system (9) for recovering the coefficients  $\mathbf{a} = \{a_{j,\ell}\}$  in the standard basis (7) of  $V_w(\tilde{\mu})$ . As  $\tilde{\mu} \rightarrow \mu_0$ , the matrix of this linear system will be  $V(w(\tilde{\mu}))$  with collision pattern  $D(w(\tilde{\mu})) \neq D_0$ , and therefore its determinant will generically approach zero. This will make the determination of  $\{a_{j,\ell}\}$  ill-conditioned, and in fact some of its components will go to infinity. At the limit, however, the confluent problem is completely

well-posed since the matrix  $V(w_0, D_0)$  is non-singular. The challenge is, therefore, to make the solution to depend continuously on  $\tilde{\mu}$  by a suitable *change of basis* for  $V_w$ . So, instead of the ill-conditioned system

$$\tilde{\mu} = V(w(\tilde{\mu}), D(w(\tilde{\mu}))) \mathbf{a}(\tilde{\mu})$$

we would like to have

$$\tilde{\mu} = \tilde{V}(\tilde{\mu}) \mathbf{b}(\tilde{\mu}), \quad (10)$$

where the matrix  $\tilde{V}$  is nonsingular and depends continuously on  $\tilde{\mu}$  in the neighborhood of  $\mu_0$ .

First, we extend the well-known definition of divided finite differences to colliding configurations.

**Definition 8** Let  $w = (x_1, \dots, x_d)$  be given. For each  $m = 1, 2, \dots, d$ , denote  $w_m = (x_1, \dots, x_m)$ . According to Definition 2, let  $s_m = s(w_m)$ ,  $T(w_m) = (\tau_{1,m}, \dots, \tau_{s_m,m})$  and  $D(w_m) = (d_{1,m}, \dots, d_{s_m,m})$ . Consider the decomposition of the rational function

$$R_{w,m}(z) = \prod_{j=1}^{s_m} \frac{1}{(z - \tau_{j,m})^{d_{j,m}}}$$

into the sum of elementary fractions

$$R_{w,m}(z) = \sum_{j=1}^{s_m} \sum_{\ell=1}^{d_{j,m}} \frac{w_{j,\ell}^{(m)}}{(z - \tau_{j,m})^\ell}. \quad (11)$$

The  $m$ -th *finite difference*  $\Delta_m(w)$  is the following element of  $V_w$ :

$$\Delta_m(w) = \sum_{j=1}^{s_m} \sum_{\ell=1}^{d_{j,m}} \frac{w_{j,\ell}^{(m)}}{(\ell-1)!} \delta^{(\ell-1)}(x - \tau_{j,m}),$$

with the coefficients  $\{w_{j,\ell}^{(m)}\}$  defined by (11).

We prove the following results in [10].

**Proposition 2** *The finite difference  $\Delta_m(w)$  is a continuous section of the bundle  $\mathcal{P}_d$ . For  $w \in \mathbb{C}^d$  with pairwise distinct coordinates,  $\Delta_m(w)$  is the usual divided finite difference on the elements of  $w_m$ .*

**Theorem 3** *For each  $w \in \mathbb{C}^d$ , the collection*

$$\mathcal{B}(w) = \{\Delta_m(w)\}_{m=1}^d \subset V_w$$

*forms a basis for  $V_w$ .*

*Remark 2* Another possible way to construct a good basis  $\tilde{\mathcal{B}}(w)$  is to build the matrix  $\tilde{V}$  in (10) directly by imitating the confluence process of the Vandermonde matrices (as done in [20]), multiplying  $V$  by an appropriate “divided difference matrix”  $F(\tilde{\mu})$  (more precisely, a chain of such matrices derived from the confluence pattern). That is,  $\tilde{V} = VF$  is the new matrix for the recovery of the linear part in (10), while the new coefficient vector is  $\mathbf{b} = F^{-1}\mathbf{a}$ . Also in this case  $\tilde{V} \rightarrow V(w_0, D_0)$  as  $\tilde{\mu} \rightarrow \mu_0$ . The matrix  $F$  thus defines the corresponding change of basis from  $\{\delta_{j,\ell}(w)\}$  as in (7) to  $\tilde{\mathcal{B}}(w)$ .

Let us now consider the Prony problem in the basis  $\mathcal{B}(w)$  in some neighborhood of  $\mu_0 \in \Sigma_d$  (thus, the order of the exact solution  $(w_0, g_0) = \mathcal{PM}^{-1}(\mu_0)$  is  $d$ ). Writing the unknown  $g \in V_w$  in this basis we have

$$g = \sum_{m=1}^d \beta_m \Delta_m(w).$$

**Theorem 4 ([10])** *For  $\tilde{\mu}$  in a sufficiently small neighborhood of  $\mu_0$ , the solution*

$$\mathcal{PM}^{-1}(\tilde{\mu}) = (w(\tilde{\mu}), \{\beta_m(\tilde{\mu})\}),$$

*expressed in the basis  $\mathcal{B}(w)$  of finite differences, is provided by continuous algebraic functions of  $\tilde{\mu}$ .*

*Proof (outline)* For each  $w$  in a neighborhood of  $w_0$ , we obtain the system of equations

$$\sum_{m=1}^d \beta_m \int x^k \Delta_m(w) = \tilde{m}_k, \quad k = 0, 1, \dots, d-1. \quad (12)$$

In the process of solution, the points  $\{x_1, \dots, x_d\}$  are found as the roots of the polynomial  $Q(z)$  which appears in the denominator of  $R_{X,D,A}(z) = \frac{P(z)}{Q(z)}$ . The coefficient vector  $\mathbf{q}$  of  $Q(z)$  is provided by solving a non-degenerate linear system

$$M_d \mathbf{q} = \begin{bmatrix} m_d \\ m_{d+1} \\ \vdots \\ m_{2d-1} \end{bmatrix}.$$

Therefore,  $w = w(\tilde{\mu})$  is given by continuous algebraic functions of  $\tilde{\mu}$ . By Proposition 2, the functions

$$\nu_{k,m}(w) = \int x^k \Delta_m(w) dx$$

are continuous in  $w$ . At  $w = w_0$ , the system (12) is non-degenerate by assumption, therefore it stays non-degenerate in a small neighborhood of  $w_0$ . Thus, the coefficients  $\beta_m(w(\tilde{\mu}))$  are also continuous algebraic functions of  $\tilde{\mu}$ .  $\square$

#### 4.2 Prony Inversion near $\Sigma'$ and Lower Rank Strata

The behavior of the inversion of the Prony mapping near the unsolvability stratum  $\Sigma'$  and near the strata where the rank of  $\tilde{M}_d$  drops, turns out to be pretty complicated. In particular, in the first case at least one of the nodes tends to infinity. In the second case, depending on the way the right-hand side  $\mu$  approaches the lower rank strata, the nodes may remain bounded, or some of them may tend to infinity. In this section we provide one initial result in this direction, as well as some examples. A comprehensive description of the inversion of the Prony mapping near  $\Sigma'$  and near the lower rank strata is important both in theoretical study and in applications of Prony-like systems, and we plan to provide further results in this direction separately.

**Theorem 5** *As the right-hand side  $\mu \in \mathbb{C}^{2d} \setminus \Sigma'$  approaches a finite point  $\mu_0 \in \Sigma'$ , at least one of the nodes  $x_1, \dots, x_d$  in the solution tends to infinity.*

*Proof* By assumptions, the components  $m_0, \dots, m_{2d-1}$  of the right-hand side  $\mu = (m_0, \dots, m_{2d-1}) \in \mathbb{C}^{2d}$  remain bounded as  $\mu \rightarrow \mu_0$ . By Theorem 4, the finite differences coordinates of the solution  $\mathcal{PM}^{-1}(\mu)$  remain bounded as well. Now, if all the nodes are also bounded, by compactness we conclude that  $\mathcal{PM}^{-1}(\mu) \rightarrow \omega \in \mathcal{P}_d$ . By continuity in the distribution space (Proposition 2) we have  $\mathcal{PM}(\omega) = \mu_0$ . Hence the Prony problem with the right-hand side  $\mu_0$  has a solution  $\omega \in \mathcal{P}_d$ , in contradiction with the assumption that  $\mu_0 \in \Sigma'$ .  $\square$

As it was shown above, for a given  $\mu \in \Sigma$  (say, with pairwise different nodes) the rank of the matrix  $\tilde{M}_d$  is equal to the number of the nodes in the solution for which the corresponding  $\delta$ -function enters with a non-zero coefficients. So  $\mu$  approaches a certain  $\mu_0$  belonging to a stratum of a lower rank of  $\tilde{M}_d$  if and only if some of the coefficients  $a_j$  in the solution tend to zero. We do not analyze all the possible scenarios of such a degeneration, noticing just that if  $\mu_0 \in \Sigma'$ , i.e., the Prony problem is unsolvable for  $\mu_0$ , then Theorem 5 remains true, with essentially the same proof. So at least one of the nodes, say,  $x_j$ , escapes to infinity. Moreover, one can show that  $a_j x_j^{2d-1}$  cannot tend to zero - otherwise the remaining linear combination of  $\delta$ -functions would provide a solution for  $\mu_0$ .

If  $\mu_0 \in \Sigma$ , i.e., the Prony problem is solvable for  $\mu_0$ , all the nodes may remain bounded, or some  $x_j$  may escape to infinity, but in such a way that  $a_j x_j^{2d-1}$  tends to zero.

## 5 Resolution of Eckhoff's problem

Consider the problem of reconstructing an integrable function  $f : [-\pi, \pi] \rightarrow \mathbb{R}$  from a finite number of its Fourier coefficients

$$c_k(f) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt, \quad k = 0, 1, \dots, M.$$

It is well-known that for periodic smooth functions, the truncated Fourier series

$$\mathfrak{F}_M(f) \stackrel{\text{def}}{=} \sum_{|k|=0}^M c_k(f) e^{ikx}$$

converges to  $f$  very fast, subsequently making Fourier analysis very attractive in a vast number of applications. By the classical Jackson's and Lebesgue's theorems [27], if  $f$  has  $d$  continuous derivatives in  $[-\pi, \pi]$  (including at the endpoints) and  $f^{(d)}(x) \in Lip(R)$ , then

$$\max_{-\pi \leq x \leq \pi} |f(x) - \mathfrak{F}_M(f)(x)| \leq C(R, d) M^{-d-1} \ln M. \quad (13)$$

Yet many realistic phenomena exhibit discontinuities, in which case the unknown function  $f$  is only piecewise-smooth. As a result, the trigonometric polynomial  $\mathfrak{F}_M(f)$  no longer provides a good approximation to  $f$  due to the slow convergence of the Fourier

series (one of the manifestations of this fact is commonly known as the ‘‘Gibbs phenomenon’’). It has very serious implications, for example when using spectral methods to calculate solutions of PDEs with shocks. Therefore an important question arises: ‘‘Can such piecewise-smooth functions be reconstructed from their Fourier measurements, with accuracy which is comparable to the ‘classical’ one (13)’?’’

It has long been known that the key problem for Fourier series acceleration is the detection of the shock locations. Applying elementary considerations we have the following fact [5].

**Proposition 3** *Let  $f$  be piecewise  $d$ -smooth. Then no deterministic algorithm can restore the locations of the discontinuities from the first  $M$  Fourier coefficients with accuracy which is asymptotically higher than  $M^{-d-2}$ .*

Let us first briefly describe what has become known as the Eckhoff’s method for this problem [16, 17, 18].

Assume that  $f$  has  $K > 0$  jump discontinuities  $\{\xi_j\}_{j=1}^K$  (they can be located also at  $\pm\pi$ , but not necessarily so). Furthermore, we assume that  $f \in C^d$  in every segment  $(\xi_{j-1}, \xi_j)$ , and we denote the associated jump magnitudes at  $\xi_j$  by

$$a_{\ell,j} \stackrel{\text{def}}{=} f^{(\ell)}(\xi_j^+) - f^{(\ell)}(\xi_j^-).$$

We write the piecewise smooth  $f$  as the sum  $f = \Psi + \Phi$ , where  $\Psi(x)$  is smooth and periodic and  $\Phi(x)$  is a piecewise polynomial of degree  $d$ , uniquely determined by  $\{\xi_j\}, \{a_{\ell,j}\}$  such that it ‘‘absorbs’’ all the discontinuities of  $f$  and its first  $d$  derivatives. This idea is very old and goes back at least to A.N.Krylov ([26]). Eckhoff derives the following explicit representation for  $\Phi(x)$ :

$$\begin{aligned} \Phi(x) &= \sum_{j=1}^K \sum_{\ell=0}^d a_{\ell,j} V_\ell(x; \xi_j) \\ V_n(x; \xi_j) &= -\frac{(2\pi)^n}{(n+1)!} B_{n+1}\left(\frac{x - \xi_j}{2\pi}\right) \quad \xi_j \leq x \leq \xi_j + 2\pi \end{aligned} \quad (14)$$

where  $V_n(x; \xi_j)$  is understood to be periodically extended to  $\mathbb{R}$  and  $B_n(x)$  is the  $n$ -th Bernoulli polynomial. Elementary integration by parts gives the following formula.

**Proposition 4** *Let  $\Phi(x)$  be given by (14). For definiteness, let us assume that  $c_0(\Phi) = \int_{-\pi}^{\pi} \Phi(x) dx = 0$ . Then*

$$c_k(\Phi) = \frac{1}{2\pi} \sum_{j=1}^K e^{-ik\xi_j} \sum_{\ell=0}^d (ik)^{-\ell-1} a_{\ell,j}, \quad k = 1, 2, \dots \quad (15)$$

Eckhoff observed that if  $\Psi$  is sufficiently smooth, then the contribution of  $c_k(\Psi)$  to  $c_k(f)$  is negligible **for large**  $k$ , and therefore one can hope to reconstruct the unknown parameters  $\{\xi_j, a_{\ell,j}\}$  from the perturbed equations (15), where the left-hand side reads  $c_k(f) \sim c_k(\Phi)$  and  $k \gg 1$ . His proposed method was to construct from the known values

$$\{c_k(f)\} \quad k = M - (d+1)K + 1, M - (d+1)K + 2, \dots, M$$

an algebraic equation satisfied by the jump points  $\{\xi_1, \dots, \xi_K\}$ , and solve this equation numerically. Based on some explicit computations for  $d = 1, 2$ ;  $K = 1$  and large number of numerical experiments, he conjectured that his method would reconstruct the jump locations with accuracy  $M^{-d-1}$ .

We consider the following generalized formulation (without referring to a specific method).

*Conjecture 1 (Eckhoff's conjecture)* The jump locations of a piecewise-smooth  $C^d$  function can be reconstructed from its first  $M$  Fourier coefficients with asymptotic accuracy  $M^{-d-2}$ .

In [11] we proposed a reconstruction method (see Algorithm 1 below) which is based on the original Eckhoff's procedure.

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**Algorithm 1** Half-order algorithm, [11].

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Let  $f \in PC(d, K)$ , and assume that  $f = \Phi^{(d)} + \Psi$  where  $\Phi^{(d)}$  is the piecewise polynomial absorbing all discontinuities of  $f$ , and  $\Psi \in C^d$ . Assume in addition the following a-priori bounds:

1. Minimal separation,  $\min_{i \neq j} |\xi_i - \xi_j| \geq J > 0$ .
2. Upper bound on jump magnitudes,  $|a_{l,j}| \leq A < \infty$ .
3. Lower bound on the value of the lowest-order jump,  $|a_{0,j}| \geq B > 0$ .
4. Upper bound on the size of the Fourier coefficients of  $\Psi$ ,  $|c_k(\Psi)| \leq R \cdot k^{-d-2}$ .

Let us be given the first  $3M$  Fourier coefficients of  $f$  for  $M > M(d, K, J, A, B, R)$  (a quantity which is computable). The reconstruction is as follows.

1. Obtain first-order approximations to the jump locations  $\{\xi_1, \dots, \xi_K\}$  by Prony's method (Eckhoff's method of order 0).
2. Localize each discontinuity  $\xi_j$  by calculating the first  $M$  Fourier coefficients of the function  $f_j = f \cdot h_j$  where  $h_j$  is a  $C^\infty$  bump function satisfying
  - (a)  $h_j \equiv 0$  on the complement of  $[\xi_j - J, \xi_j + J]$ ;
  - (b)  $h_j \equiv 1$  on  $[\xi_j - \frac{J}{3}, \xi_j + \frac{J}{3}]$ .
3. Fix the reconstruction order  $d_1 \leq \lfloor \frac{d}{2} \rfloor$ . For each  $j = 1, 2, \dots, K$ , recover the parameters  $\{\xi_j, a_{0,j}, \dots, a_{d_1,j}\}$  from the  $d_1 + 2$  equations

$$c_k(f_j) = \frac{1}{2\pi} e^{-i\xi_j k} \sum_{l=0}^{d_1} \frac{a_{l,j}}{(ik)^{l+1}}, \quad k = M - d_1 - 1, M - d_1, \dots, M$$

by Eckhoff's method for one jump (in this case we get a single polynomial equation  $\{p_M^{d_1}(\xi_j) = 0\}$  of degree  $d_1$ ).

4. From the previous steps we obtained approximate values for the parameters  $\{\tilde{\xi}_j\}$  and  $\{\tilde{a}_{l,j}\}$ . The final approximation is taken to be

$$\tilde{f} = \tilde{\Psi} + \tilde{\Phi} = \sum_{|k| \leq M} \left\{ c_k(f) - \frac{1}{2\pi} \sum_{j=1}^K e^{-i\tilde{\xi}_j k} \sum_{l=0}^{d_1} \frac{\tilde{a}_{l,j}}{(ik)^{l+1}} \right\} e^{ikx} + \sum_{j=1}^K \sum_{l=0}^{d_1} \tilde{a}_{l,j} V_l(x; \tilde{\xi}_j).$$


---

We have also shown that this method achieves the following accuracy.

**Theorem 6 ([11])** *Let  $f \in PC(d, K)$  and let  $\tilde{f}$  be the approximation of order  $d_1 \leq \lfloor \frac{d}{2} \rfloor$  computed by Algorithm 1. Then<sup>1</sup>*

$$\begin{aligned} \left| \tilde{\xi}_j - \xi_j \right| &\leq C_1(d, d_1, K, J, A, B, R) \cdot M^{-d_1-2}; \\ \left| \tilde{a}_{l,j} - a_{l,j} \right| &\leq C_2(d, d_1, K, J, A, B, R) \cdot M^{l-d_1-1}, \quad l = 0, 1, \dots, d_1; \\ \left| \tilde{f}(x) - f(x) \right| &\leq C_3(d, d_1, K, J, A, B, R) \cdot M^{-d_1-1}. \end{aligned} \quad (16)$$

The non-trivial part of the proof of this result was to analyze in detail the polynomial equation  $p(\xi_j) = 0$  in step 3 of Algorithm 1. It turned out that additional orders of smoothness (namely, between  $d_1$  and  $d$ ) produced an error term which, when substituted into the polynomial  $p$ , resulted in unexpected cancellations due to which the root  $\xi_j$  was perturbed only by  $O(M^{-d_1})$ . This phenomenon was first noticed by Eckhoff himself in [17] for  $d = 1$ , but at the time its full significance was not realized.

An important property of Algorithm 1 is that its final asymptotic convergence order essentially depends on the accuracy of step 3. It is sufficient therefore to replace this step with another method which achieves full accuracy (i.e.  $\sim M^{-d-2}$ ) in order to obtain the overall reconstruction with full accuracy. It turns out that taking instead of consecutive Fourier samples

$$k = M - d - 1, M - d, \dots, M$$

the “decimated” section

$$k = N, 2N, \dots, (d+2)N; \quad N \stackrel{\text{def}}{=} \left\lfloor \frac{M}{(d+2)} \right\rfloor$$

provides this accuracy.

For the full details, see [5]. Here let us outline our method of proof.

Denote the single jump point by  $\xi \in [-\pi, \pi]$ , and let  $\omega = e^{-i\xi}$ . The purpose is to recover the jump point  $\omega$  and the jump magnitudes  $\{a_0, \dots, a_d\}$  from the noisy measurements

$$\tilde{c}_k(f) = \underbrace{\frac{\omega^k}{2\pi} \sum_{j=0}^d \frac{a_j}{(ik)^{j+1}}}_{\stackrel{\text{def}}{=} c_k} + \epsilon_k, \quad k = N, 2N, \dots, (d+2)N, \quad |\epsilon_k| \leq R \cdot k^{-d-2}. \quad (17)$$

Again, we multiply (17) by  $(2\pi)(ik)^{d+1}$ . Denote  $\alpha_j = i^{d+1-j} a_{d-j}$ . We get

$$\tilde{m}_k \stackrel{\text{def}}{=} 2\pi(ik)^{d+1} \tilde{c}_k = \omega^k \underbrace{\sum_{j=0}^d \alpha_j k^j}_{\stackrel{\text{def}}{=} m_k} + \delta_k, \quad k = N, 2N, \dots, (d+2)N, \quad |\delta_k| \leq R \cdot k^{-1}. \quad (18)$$

<sup>1</sup> The last (pointwise) bound holds on “jump-free” regions.

**Definition 9** Let

$$p_N^d(u) \stackrel{\text{def}}{=} \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} m_{(j+1)N} u^{d+1-j}.$$

**Proposition 5** The point  $u = \omega^N$  is a root of  $p_N^d(u)$ .

**Proposition 6** The vector of exact magnitudes  $\{\alpha_j\}$  satisfies

$$\begin{bmatrix} m_N \omega^{-N} \\ m_{2N} \omega^{-2N} \\ \vdots \\ m_{(d+1)N} \omega^{-(d+1)N} \end{bmatrix} = \begin{bmatrix} 1 & N & N^2 & \dots & N^d \\ 1 & 2N & (2N)^2 & \dots & (2N)^d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & (d+1)N & ((d+1)N)^2 & \dots & ((d+1)N)^d \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_d \end{bmatrix}. \quad (19)$$

The procedure for recovery of the  $\{\alpha_0, \dots, \alpha_d, \omega\}$  is presented in Algorithm 2 below, while the method for full recovery of the function is outlined in Algorithm 3 below.

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**Algorithm 2** Recovery of single jump parameters

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Let there be given the first  $N \gg 1$  Fourier coefficients of the function  $f_j$ , and assume that the jump position  $\xi$  is already known with accuracy  $o(N^{-1})$ .

1. Construct the polynomial

$$q_N^d(u) = \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} \tilde{m}_{(j+1)N} u^{d+1-j}.$$

from the given noisy measurements  $\tilde{m}_N, \tilde{m}_{2N}, \dots, \tilde{m}_{(d+2)N}$  (18).

2. Find the root  $\tilde{z}$  which is closest to the unit circle (in fact any root will do).
3. Take  $\tilde{\omega} = \sqrt[N]{\tilde{z}}$ . Note that in general there are  $N$  possible values on the unit circle (see Remark 3), but since we already know the approximate location of  $\omega$  the correct value can be chosen consistently.
4. Recover  $\tilde{\xi} = -\arg \tilde{\omega}$ .
5. To recover the magnitudes, solve the linear system (19).

*Remark 3* To see that there are  $N$  possible solutions, notice that one recovers  $e^{i\xi N} = e^{it}$ , which is satisfied by any  $\xi = \frac{t}{N} + \frac{2\pi}{N}n$ ,  $n \in \mathbb{Z}$  and not just  $\xi = \frac{t}{N}$ .

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**Algorithm 3** Full accuracy Fourier approximation

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Let  $f \in PC(d, K)$ , and assume that  $f = \Phi^{(d)} + \Psi$  where  $\Phi^{(d)}$  is the piecewise polynomial absorbing all discontinuities of  $f$ , and  $\Psi \in C^d$ . Assume the a-priori bounds as in Algorithm 1.

1. Using Algorithm 1, obtain approximate values of the jumps up to accuracy  $O(N^{-\lfloor \frac{d}{2} \rfloor - 2}) = o(N^{-1})$ , and the Fourier coefficients of the functions  $f_j$ .
  2. Use Algorithm 2 to further improve the accuracy of reconstruction.
-

We have shown that indeed full accuracy is achieved.

**Theorem 7 ([5])** *Algorithm 2 recovers the parameters of a single jump from the given noisy measurements (18) with the following accuracy:*

$$\begin{aligned} |\tilde{\xi} - \xi| &\leq C_4 \frac{R}{B} N^{-d-2}, \\ |\tilde{\alpha}_j - \alpha_j| &\leq C_5 R \left(1 + \frac{A}{B}\right) N^{-j-1}, \quad j = 0, 1, \dots, d. \end{aligned}$$

The main idea of the proof is to analyze the perturbation of the polynomial  $p_N^d(u)$  by  $q_N^d$  using Rouché's theorem.

After making the substitution  $N = \lfloor \frac{M}{(d+2)} \rfloor$ , we obtain as an immediate consequence of Theorem 7 the resolution of Conjecture 1.

**Theorem 8** *Let  $f \in PC(d, K)$  and let  $\tilde{f}$  be the approximation of order  $d$  computed by Algorithm 3. Then*

$$\begin{aligned} |\tilde{\xi}_j - \xi_j| &\leq C_6(d, K, J, A, B, R) \cdot M^{-d-2}, \\ |\tilde{a}_{l,j} - a_{l,j}| &\leq C_7(d, K, J, A, B, R) \cdot M^{l-d-1}, \quad l = 0, 1, \dots, d; \\ |\tilde{f}(x) - f(x)| &\leq C_8(d, K, J, A, B, R) \cdot M^{-d-1}. \end{aligned} \quad (20)$$

Note that the system (17) is a certain variant of the confluent Prony system (5) for just one node. Therefore, Algorithm 2 can be regarded as a concrete solution method for this particular case.

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