Vandermonde matrices

$$\mathbf{V}_N = \begin{bmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_R \\ z_1^2 & z_2^2 & \dots & z_R^2 \\ \vdots & \vdots & \vdots & \vdots \\ z_1^N & z_2^N & \dots & z_R^N \end{bmatrix}$$

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- $z_j \in \mathbb{R}$: polynomial interpolation/approximation, quadrature, moment problems,...
- $z_j = e^{\imath x_j}$, $x_j \in (-\pi, \pi]$: Fourier analysis, super-resolution

Super-resolution

- Unknown signal

$$f(x) = \sum_{j=1}^{R} \alpha_j \delta_{x_j}, \ \alpha_j \in \mathbb{C} \backslash \{0\}, \ x_j \in \mathbb{R}^d \quad (d = 1)$$

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$$m_k = \sum_{j=1}^{R} \alpha_j e^{\imath k x_j} + n_k, \quad k = 0, 1, \ldots, N-1;$$

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- **Stability?**

Applications

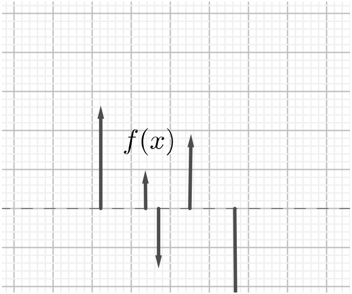
- Inverse problems with sparse priors
- Super-resolution Imaging
- Sub-Nyquist/Finite Rate of Innovation sampling
- Direction of Arrival estimation
- Exponential data analysis / fitting
- Line spectral estimation / "harmonic inversion"
- Compressed sensing
- Mixture models
- Pade approximation, moment problems, sparse interpolation, sparse FFT, analytic number theory,...

T.Bendory, J.J.Benedetto, A.Bhandari, T.Blu, H.Bölcskei, E.Candes, A.Cuyt, L.Demanet, D.Donoho, P-L.Dragotti, A.Fannjiang, C.Fernandez-Granda, F.Filbir, P.Indyk, S.Kunis, W-s. Lee, W.Li, W.Liao, H.Mhaskar, A.Moitra, V.Morgenshtern, G.Peyre, G.Plonka, C.Poon, D.Potts, P.Stoica, M.Tasche, M.Vetterli,...

Stability of sparse SR

$$f(x) = \sum_{j=1}^{R} \alpha_j \delta_{x_j}, \mathbf{m} = \mathbf{V}_N(x_1, \ldots, x_R)\boldsymbol{\alpha} + \mathbf{n}$$

$$x_j \in \Delta\mathbb{Z} \quad (\text{“on-grid”})$$



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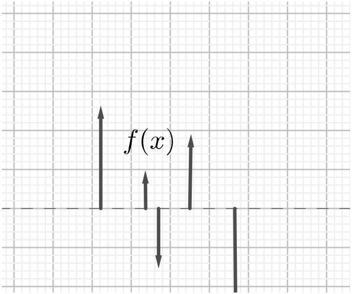
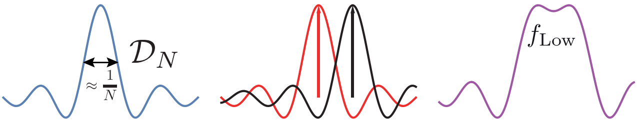


Lemma (Min-max rate and $\mathbf{V}_N$)

$$\inf_{\widetilde{f}} \sup_{f} \sup_{\|n\|_2 \leqslant \varepsilon} \|\widetilde{f} - f\|_2 \asymp \min_{\mathbf{y} \in \mathbb{R}^{2R}} \frac{\varepsilon}{\sigma_{\min}(\mathbf{V}_N(\mathbf{y}))}$$

Classical resolution limit



- High resolution signal: $f_{\mathrm{High}}(x) = \sum_j \alpha_j \delta_{x_j}$
- Dirichlet kernel:

$$\mathcal{D}_N(x) = \sum_{k=-N}^{N} \exp(\imath k x) = \begin{cases} \frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{\sin \frac{x}{2}} & x \notin 2\pi\mathbb{Z} \\ 2N+1 & \text{else} \end{cases}$$

- Low resolution signal:

$$f_{\mathrm{Low}}(x) = \sum_{|k| \leqslant N} m_k e^{-\imath k x} \asymp \sum_j \alpha_j \mathcal{D}_N(x - x_j)$$

Rayleigh-Nyquist length $\asymp \frac{1}{N}$

$\sigma_{\min}$: extreme cases

$$\Delta := \min_{j \neq k} |x_j - x_k|$$

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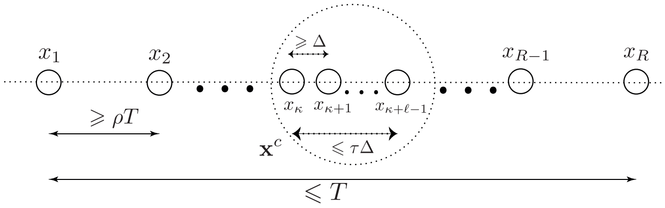
- **Can we do better?**

Partial clustering

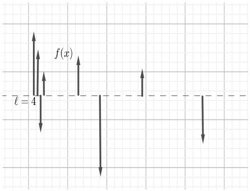
References

1. D. Batenkov and G. Goldman, "Single-exponential bounds for the smallest singular value of Vandermonde matrices in the sub-Rayleigh regime," **Applied and Computational Harmonic Analysis, 2021.**
2. D. Batenkov, G. Goldman, and Y. Yomdin, "Super-resolution of near-colliding point sources," **Inf Inference, 2021.**
3. D. Batenkov, B. Diederichs, G. Goldman, and Y. Yomdin, "The spectral properties of Vandermonde matrices with clustered nodes," **Linear Algebra and its Applications, 2021.**
4. D. Batenkov, L. Demanet, G. Goldman, and Y. Yomdin, "Conditioning of Partial Nonuniform Fourier Matrices with Clustered Nodes," **SIAM J. Matrix Anal. Appl., 2020**.

Partial clustering model



- "Cluster":= nodes closer than $\frac{1}{N}$
- Geometric "rigidity" parameters ρ, τ
- Can rescale the support to $[0, 1]$



How to estimate $\sigma_{\min}(V_N)$?

Spectral properties of Fourier-type matrices

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Assumptions: $N\Delta \ll 1,\ N \gg 1$

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The column subspaces corresponding to the different clusters are nearly orthogonal: $\angle_{\min}(C_i, C_j) \geqslant \frac{\pi}{2} - \frac{c_1}{N\rho} - c_2 N\Delta$.

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3. Other methods: Diederichs 2019, Li&Liao 2021, Kunis&Nagel 2020a,b

Single cluster estimates

Estimating *all* the singular values of C_i

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- $\mathcal{E}_{2m}$ are PSD restricted to the kernel of $\mathbf{V}_m(x_1, \ldots, x_R)$ (real Vandermonde) [Micchelli 1986]

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$\rightsquigarrow$ Nikolskii-type inequality [Erdélyi 2017]:

$$\|P\|_{L^p[0,1]} \leqslant \left(\frac{\pi\ell}{2}\right)^{2/q-2/p} \|P\|_{L^q[0,1]}, \quad 0 < q < p \leqslant \infty, \quad q \leqslant 2.$$

Implications for super-resolution

On-grid minimax recovery

$$f = \sum_{j=1}^{R} \alpha_j \delta_{x_j}, \quad x_j \in \Delta\mathbb{Z} \cap \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\hat{g}(\omega) = \sum_{j=1}^{R} \alpha_j e^{\imath \omega x_j} + n(\omega), \quad |\omega| \leqslant \Omega, \; \|n\|_{2,\Omega} \leqslant \varepsilon.$$

Theorem (SIMA 2020)

In the on-grid model with clusters of size at most ℓ:

$$\inf_{\widetilde{f}} \sup_{f} \sup_{\|n\|_2 \leqslant \varepsilon} \|\widetilde{f} - f\|_2 \asymp SRF^{2\ell-1}\varepsilon, \quad SRF := \frac{1}{\Omega\Delta}.$$

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- [Morgenshtern&Candes 2016] $\alpha_j > 0$, ℓ_1 recovery gives $\leqslant \mathrm{SRF}^{2\ell}\varepsilon$

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Definition (Minimax rate)

$$\Lambda^{\mathbf{x},j}(\varepsilon, U, \Omega) = \inf_{\widetilde{f}=(\widetilde{\boldsymbol{\alpha}},\widetilde{\mathbf{x}})} \; \sup_{f=(\boldsymbol{\alpha},\mathbf{x})\in U} \; \sup_{n:\ \|n\|_\infty \leqslant \varepsilon} \|\mathbf{x}_j - \widetilde{\mathbf{x}}_j\|,$$

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Theorem (Inf.Inference 2021)

For $SRF := \frac{1}{\Omega\Delta} \geqslant O(1)$, $\{\alpha_j\}$ bdd., cluster $\mathbf{x}^c$ of size $\ell \leqslant R$, $\varepsilon \lessapprox (\Omega\Delta)^{2\ell-1}$:

$$\Lambda^{\mathbf{x},j}(\varepsilon, U, \Omega) \asymp \begin{cases} SRF^{2\ell-1}\Delta\varepsilon & x_j \in \mathbf{x}^c, \\ \frac{\varepsilon}{\Omega} & x_j \in \mathbf{x}\setminus\mathbf{x}^c, \end{cases}$$

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Open problems

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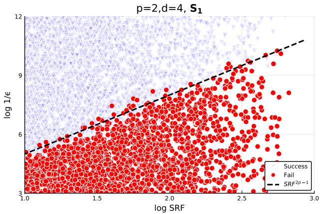

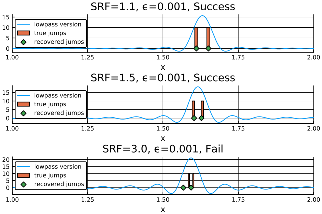

- Provably optimal and tractable algorithms

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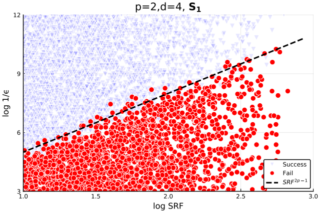

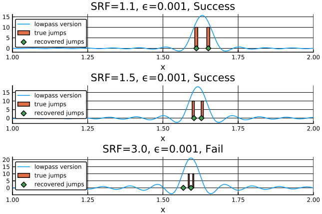

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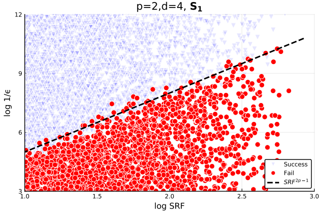
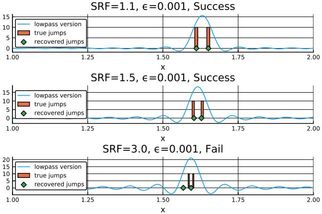
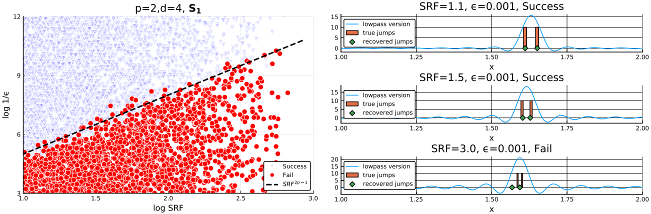




- Provably optimal and tractable algorithms
- Optimal constants in minimax (can we have $R \to \infty$ with ℓ small?)
- Stable recovery of information when $\varepsilon \gtrapprox (\Omega\Delta)^{2\ell'-1}$ with $\ell' < \ell$: need to understand singular vectors

Open problems



- Provably optimal and tractable algorithms
- Optimal constants in minimax (can we have $R \to \infty$ with ℓ small?)
- Stable recovery of information when $\varepsilon \gtrsim (\Omega\Delta)^{2\ell'-1}$ with $\ell' < \ell$: need to understand singular vectors
- High-order models: $U_N(\mathbf{x}) = \underbrace{\left[V_N(\mathbf{x}) \; V_N'(\mathbf{x})\right]_{N\times 2R}}_{\text{Confluent Vandermonde matrix}}$

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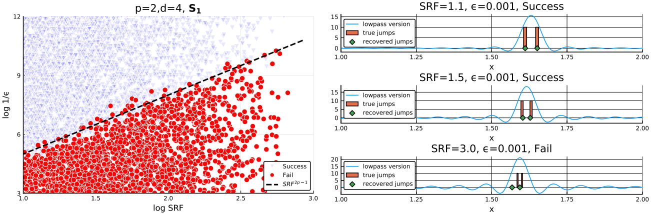

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- $d > 1$

High-order distributions

$$f(x) = \sum_{j=1}^{R} (\alpha_j \delta_{x_j} + \beta_j \delta'_{x_j}), \quad x_j \in \Delta\mathbb{Z} \cap \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \|f\|_2^2 = \sum_j |\alpha_j|^2 + |\beta_j|^2$$

$\Rightarrow$ Algebraic recovery (quadrature domains, piecewise-smooth functions,...)

Theorem (DB and Diab, arxiv 2203.11923)

In the ℓ-clustered model and discrete measurements:

$$\inf_{\tilde{f}} \sup_{f} \sup_{\|noise\|_2 \leqslant \varepsilon} \|\tilde{f} - f\|_2 \asymp SRF^{4\ell-1}\varepsilon, \quad SRF \to \infty.$$

$\rightsquigarrow$ *confluent* Vandermonde matrices $U_N(\mathbf{x}) = [V_N(\mathbf{x}) \ V'_N(\mathbf{x})]_{N \times 2R}$

- Lower bound: further technical restriction on the support
- Upper bound on $\sigma_{\min}(U_N(\mathbf{x}))$ holds *for any* $\mathbf{x}$

High-order distributions

$$f(x)=\sum_{j=1}^{R}(\alpha_j\delta_{x_j}+\beta_j\delta'_{x_j}),\quad x_j\in\Delta\mathbb{Z}\cap\left[-\frac{\pi}{2},\frac{\pi}{2}\right],\|f\|_2^2=\sum_j|\alpha_j|^2+|\beta_j|^2$$

$$\hat{f}(\omega)=\sum_{j=1}^{R}(\alpha_j-\imath\omega\beta_j)e^{\imath\omega x_j},\qquad \omega\in[-\Omega,\Omega]$$

$$U_N(\mathbf{x})=\left[V_N(\mathbf{x})\ V'_N(\mathbf{x})\right]_{N\times 2R}\qquad \text{(confluent Vandermonde)}$$

Theorem (B, Diab, **arxiv 2203.11923**)

$$\sigma_{\min}(U_N)\asymp SRF^{1-2\ell}$$

$$\inf_{\tilde{f}}\sup_{f}\sup_{\|noise\|_2\leqslant\varepsilon}\|\tilde{f}-f\|_2\asymp SRF^{4\ell-1}\varepsilon,\quad SRF\gg 1.$$

- Worst-case amplitude construction *for any* $\mathbf{x}$

References

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