Lecture 9

Parametric Super-Resolution

Topics in Inverse Problems and Super-Resolution

Dr. Dmitry Batenkov

Fall 2022

Some recap

- "Classical SR"
  - Linear measurement model: $y^{\delta} = P_{\Omega}x + e, \quad \|e\| \leq \delta$
  - Support constraint: $\operatorname{supp} x \subset [-T, T] \implies \hat{x} \in PW_T$ **- Linear!**
  - Linear method: regularized sol., i.e. $x^{\delta} = R_{\delta,\alpha}y^{\delta}$
  - log. stability: $\|x - x^{\delta}\| \sim \log^{\beta} 1/\delta$
- ℓ_1 minimization - "Nonparametric" approach
  - Linear measurement model: $y = F_n x$
  - Nonlinear sparsity constraint: $\|x\|_0 \leq K$
  - Nonlinear method: $x^* = \arg\min_x \|x\|_1$ s.t. $\|F_n x - y^{\delta}\| \leq \delta$
  - Linear stability: $\|x - x^*\| \sim \mathrm{SRF}^{2r}\,\delta$
  - Restrictions: positivity/separation

$$x = \sum_j a_j \delta_{t_j}$$

EXPERIMENTAL AND ANALYTICAL ESSAY
ON
THE EXPANSION PROPERTIES OF ELASTIC FLUIDS
AND ON THE FORCE OF EXPANSION OF
WATER VAPOR AND ALCOHOL VAPOR AT DIFFERENT
TEMPERATURES

By R. Prony, 1795
Translated by Dr. Ann Sanders/ETI

Gaspard-Clair-Francois-Marie
Riche de Prony
(1755-1839)

Prony's problem: recover $\{\mu_j, \rho_j\}$ from samples of

$$\tau(x) = \mu_1 \rho_1^x + \mu_2 \rho_2^x + \cdots \mu_n \rho_n^x$$

Gaspard Riche de Prony, *Essai experimental et analytique: sur les lois de la dilatabilite de fluides elastique et sur celles de la force expansive de la vapeur de l'alkool, a differentes temperatures*. **J. Ecole Polyt. 1:24-76 (1795).**

Prony's Insight

$$\tau(k) = \sum_{j=1}^{n} \mu_j \rho_j^k , \qquad \mu_j \neq 0, \rho_j \in \mathbb{C}$$

$$Q(x) = (x - \rho_1) \cdots (x - \rho_n) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

$$\Rightarrow \sum_{\ell=0}^{n} c_\ell \tau(k + \ell) = 0, \qquad \forall k \in \mathbb{N}$$

Proof: $\sum_{\ell=0}^{n} c_\ell \tau(k+\ell) = \sum_{\ell=0}^{n} c_\ell \sum_{j=1}^{n} \mu_j \rho_j^{k+\ell} = \sum_{j=1}^{n} \mu_j \rho_j^k \, Q(\rho_j) = 0$

Prony's Method

$$\tau(k) = \sum_{j=1}^{n} \mu_j \rho_j^k, \qquad \mu_j \neq 0, \rho_j \in \mathbb{C}$$

$2n$ unknowns

- Construct $n \times (n+1)$ **Hankel** matrix

$$H = [\tau(k+\ell)]_{k=0,1,\dots,n-1}^{\ell=0,1,\dots,n} = \begin{bmatrix} \tau(0) & \cdots & \tau(n) \\ \vdots & \ddots & \vdots \\ \tau(n-1) & \cdots & \tau(2n-1) \end{bmatrix}$$

$2n$ samples

- Find $\boldsymbol{c} \in \ker(H)$, construct $Q_{\boldsymbol{c}}(x) = \sum_{j=0}^{n} c_j x^j$
- $\{\rho_j\}$ are the roots of $Q_{\boldsymbol{c}}$
- $\{\mu_j\}$ are given by

Vandermonde matrix

$$\begin{bmatrix} 1 & \cdots & 1 \\ \rho_1 & \cdots & \rho_n \\ \vdots & \ddots & \vdots \\ \rho_1^{n-1} & \cdots & \rho_n^{n-1} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} = \begin{bmatrix} \tau(0) \\ \tau(1) \\ \vdots \\ \tau(n-1) \end{bmatrix}$$

Applications

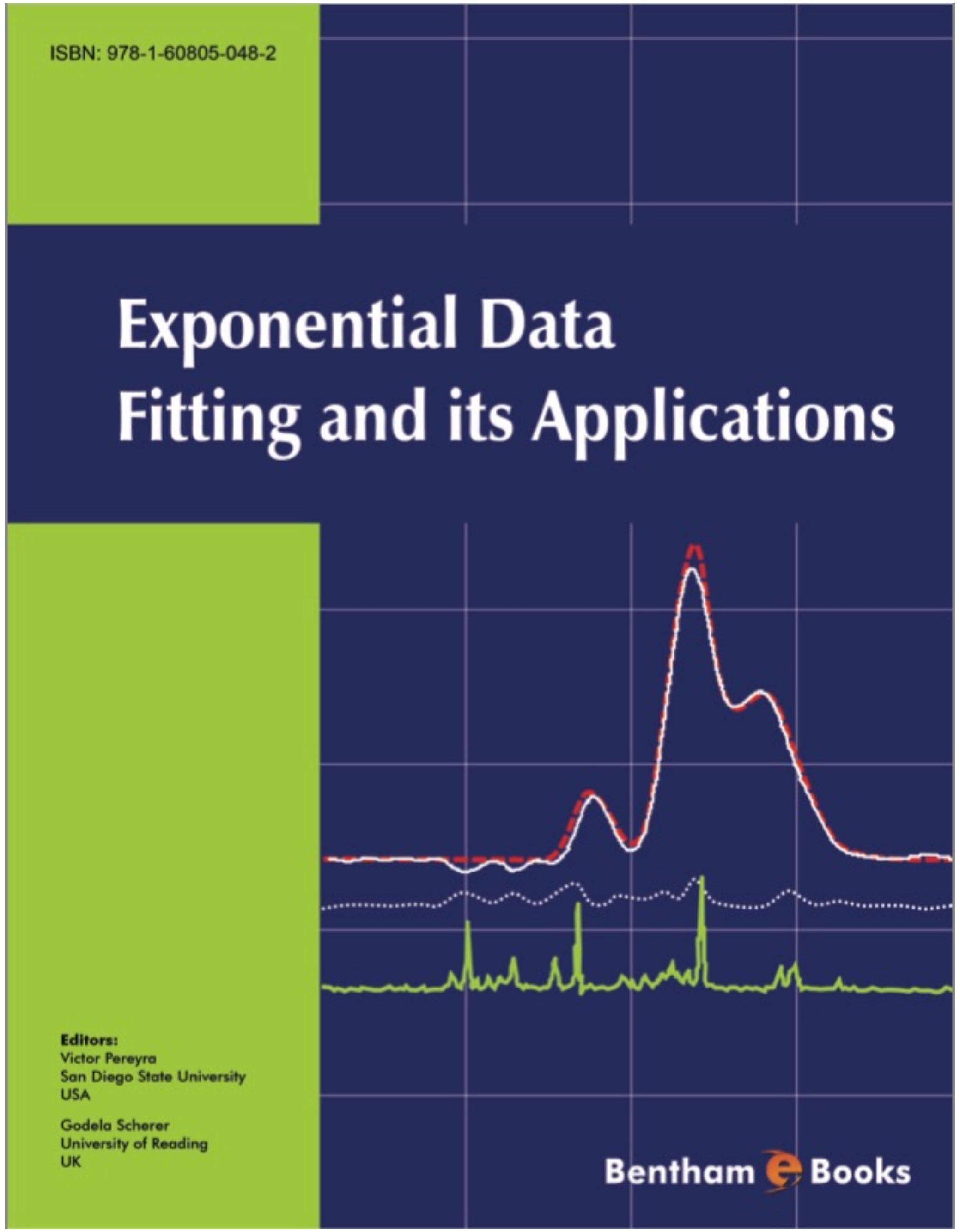
- Data analysis
  - Exponential/rational function fitting
  - Sparse interpolation
- Signal processing
  - Direction of arrival estimation
  - Parametric spectral estimation
  - Sampling below the Nyquist rate
- Inverse problems
  - Inverse scattering/source problems
  - Resolution beyond the diffraction limit
- Numerical analysis
  - Pade approximation
  - Sequence extrapolation
- …

$$\tau(x) \approx \mu_1\rho_1^x + \mu_2\rho_2^x + \cdots \mu_n\rho_n^x$$

Exponential data analysis

$$\tau(x) \approx \mu_1 e^{-\rho_1 x} + \mu_2 e^{-\rho_2 x} + \cdots \mu_n e^{-\rho_n x}$$

- Computational methods
    - Maximum Likelihood (least squares)
    - Modified Prony method
    - VARiable PROjections (VARPRO)
    - Subspace methods (HSVD)
- Applications:
    - MR spectroscopy
    - MRI
    - Quantum Field Theory
    - Time-resolved spectroscopy
    - Optically stimulated luminescence
    - Supernova Light Curves
    - VLSI computations

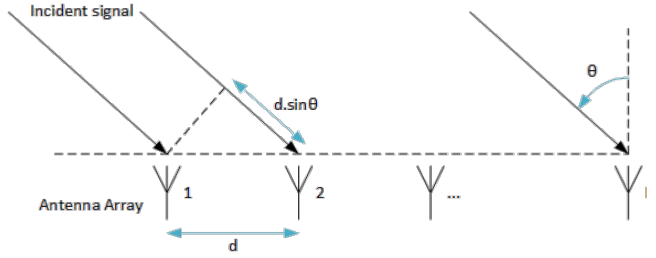


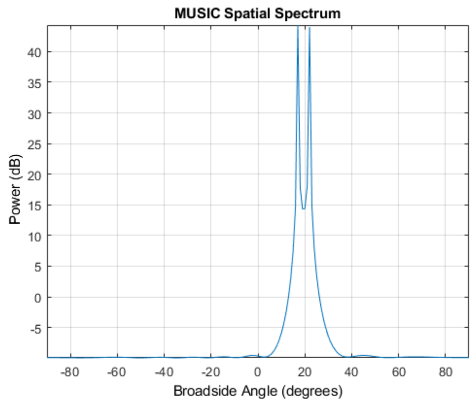
Signal processing

Direction of arrival / inverse scattering

Signal: $$f(t) \sim \sum_{j=1}^{n} \mu_j \delta(t - t_j)$$

Moments: $$\tau(k) = \mathcal{F}\{f\}(k) \sim \sum_{j=1}^{n} \mu_j e^{2\pi i t_j k}$$

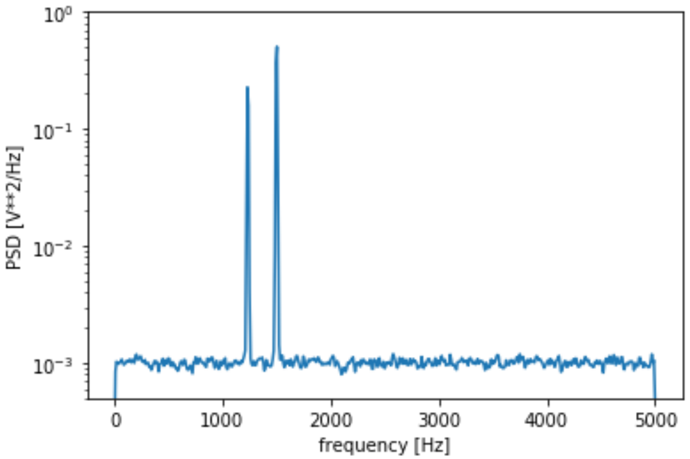




Spectral estimation/ harmonic inversion/ hidden periodicities/ …

(Line) spectrum: $$\hat{\tau}(\omega) \sim \sum_{j=1}^{n} \mu_j \delta(\omega - \omega_j)$$

Signal: $$\tau(x) \sim \sum_{j=1}^{n} \mu_j e^{i\omega_j x}$$



Beyond classical sampling

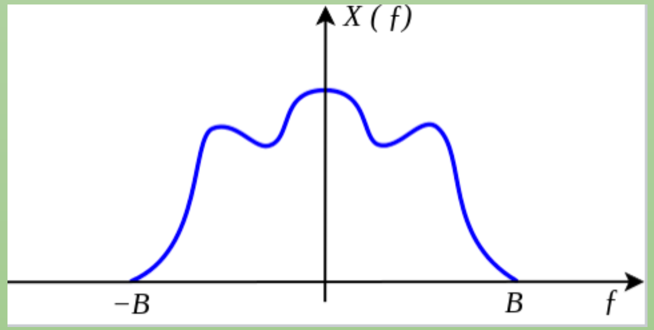
$$f(t) \sim \left(\sum_{j=1}^{n} \mu_j \delta(\cdot - t_j) * h \right)(t) = \sum_{j=1}^{n} \mu_j h(t - t_j)$$

- Samples: $\mathrm{y_n} = f(nT)$
- For many choices of h there exist coefficients $c_{m,n}$ s.t.

$$s(m) = \sum_{n} c_{m,n}(h) y_n = \sum_{j=1}^{n} \mu_j e^{\lambda m t_j}$$

- $\Rightarrow$ Can recover $f(t)$ from samples far below Nyquist rate

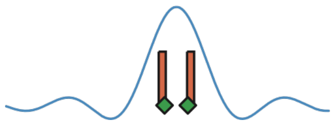
Shannon-Nyquist-Kotelnikov-Whittaker Th.



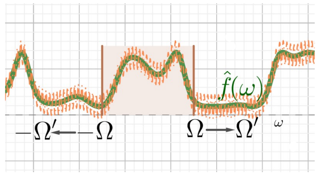
$x(t) = \sum_{n \in \mathbb{Z}} x(nT) \mathrm{sinc}\left(\frac{t-nT}{T}\right)$ holds if $T < \frac{1}{2B}$

(1) Vetterli, M.; Marziliano, P.; Blu, T. **2002**.
(2) Dragotti, P. L.; Vetterli, M.; Blu, T. **2007.**

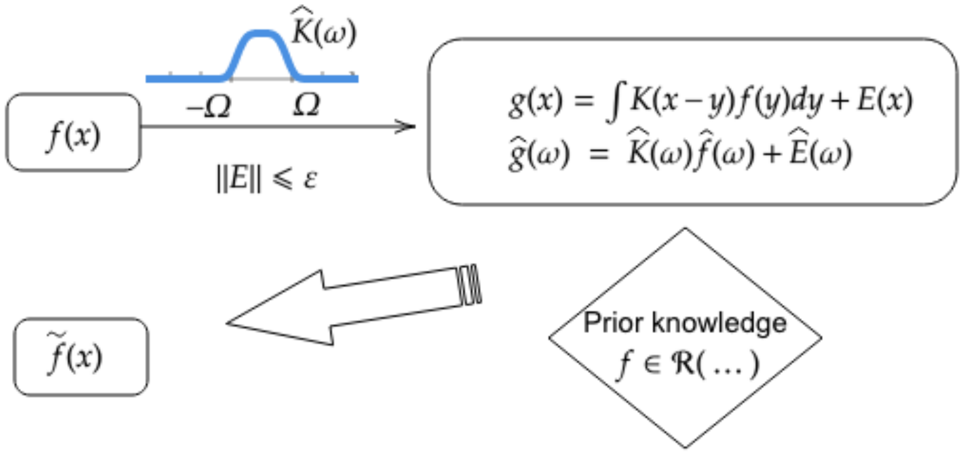
Inverse problems & Super-resolution



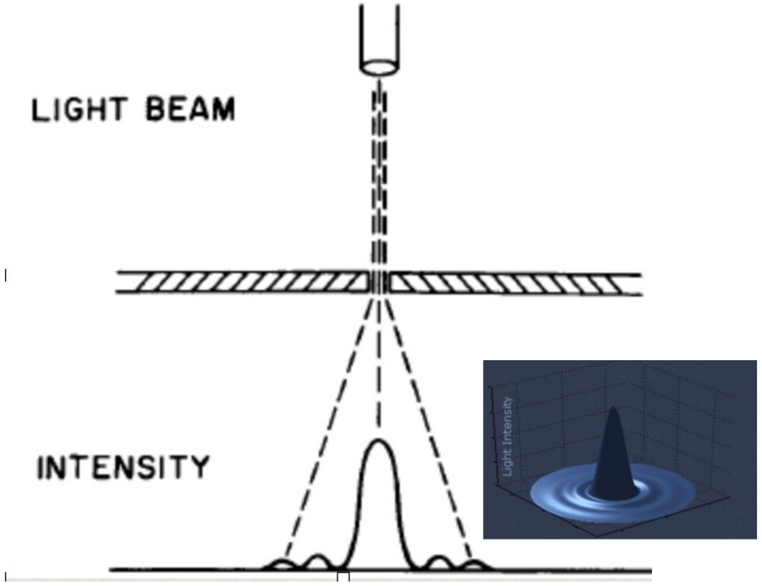
SR in time/space



SR in frequency



- Super-resolution (SR): recover $\hat{f}(\omega)$ for $|\omega| > \Omega$
- For $\mathcal{R} = \{\sum_{j=1}^{n} \mu_j \delta(t - t_j)\}$ can have unlimited SR
- Eventual resolution limit depends on
  - Noise level $||E||$
  - Prior complexity/geometry of $\{t_j\}$
  - Bandlimit Ω

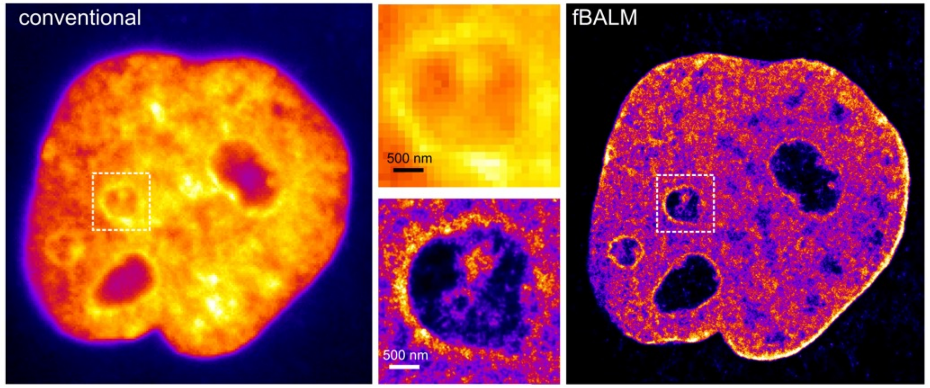


- Cameras
- Telescopes
- Microscopes

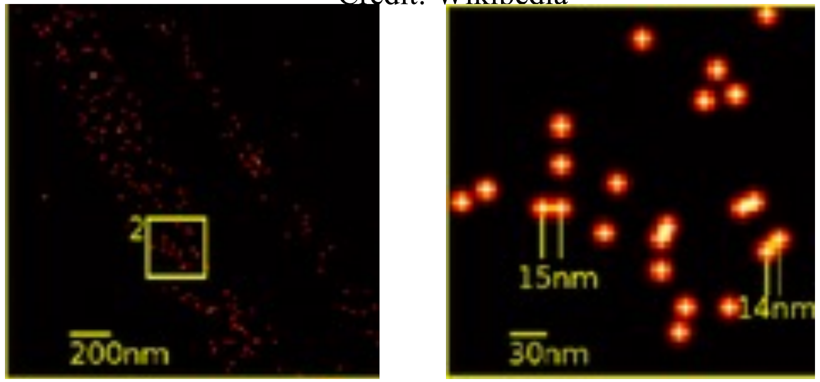
✱Diffraction-limited imaging
✱Limited sampling rate
✱Limited acquisition time
✱…

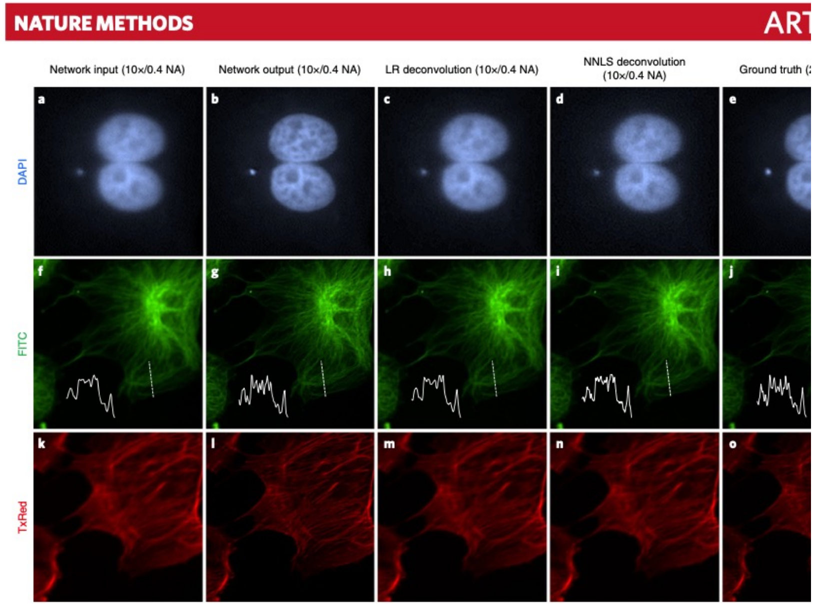
Q: how and when can we reliably extract high-res. information?

SR today

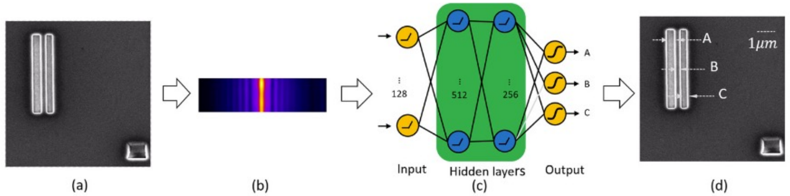


Credit: Wikipedia





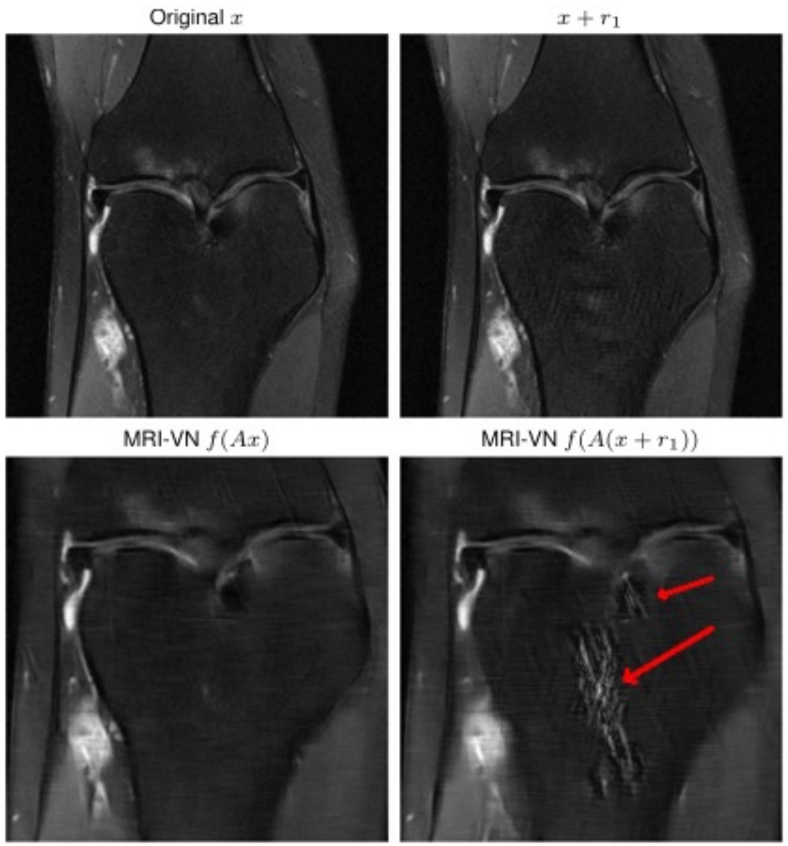
Credit: *Nat. Methods* **16**, 103–110 (2019).



Credit: *Appl. Phys. Lett.116, 131105 (2020)*

HUGE SUCCESS! …or is it??

Can we be sure?



Credit: *Proc Natl Acad Sci USA* 201907377 (2020)

* Artifacts in reconstruction
* Adversarial noise attacks

Rigorous SR guarantees largely lacking!

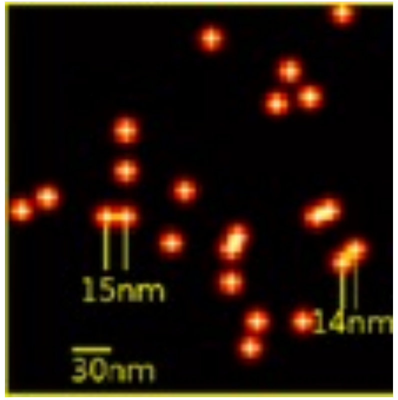
“Algebraic SR” - models

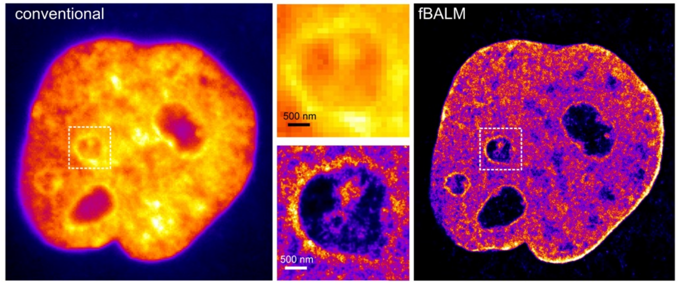
Requirements

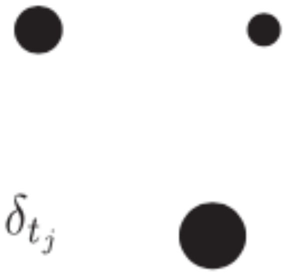
- Interpretability
- Good approximation
- Tractable analysis of resolution
- Tractable algorithms!

Semi-parametric models

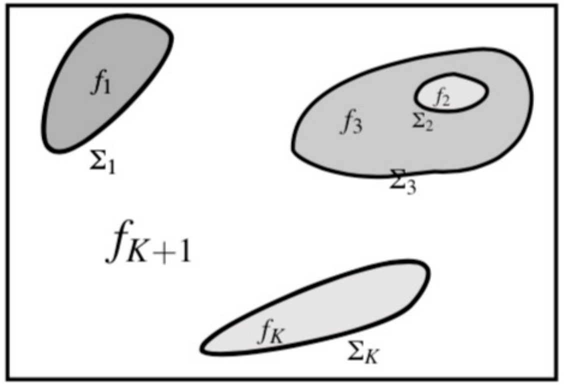
- A few explicit geometric parameters
- Restrict local complexity —> stability
- Can apply powerful math. tools

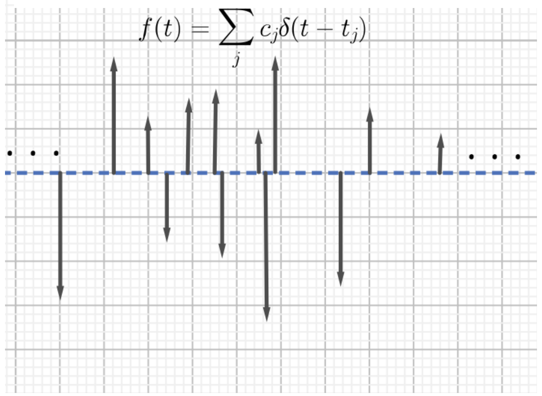


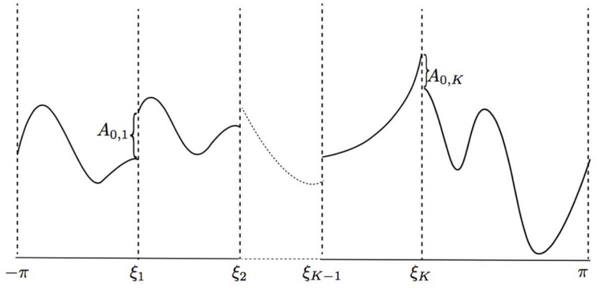




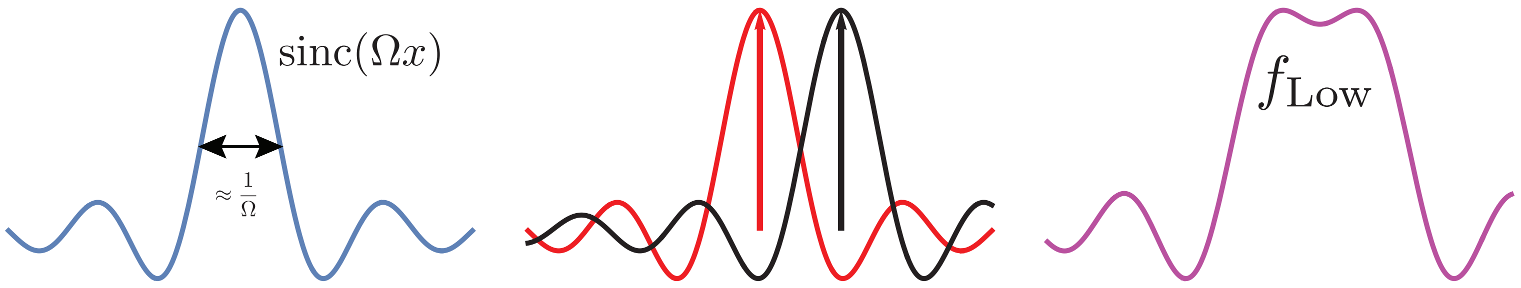
$$\mu \sim \sum_j a_j \delta_{t_j}$$







Parametric 1D SR



- High-resolution signal:

$$f_H = \sum_j \mu_j \delta(t - t_j)$$

- Low-resolution signal:

$$F_{\mathrm{Low}} \approx \sum_j \mu_j \,\mathrm{sinc}\big(\Omega(t - t_j)\big)$$

Rayleigh limit: $\sim \frac{1}{\Omega}$

- Relevant applications
  - Radio astronomy
  - Interference spectroscopy
  - Seismic data processing
  - MR spectroscopy

Parametric 1D moment problem (MP)

$$\tau(k) = \sum_{j=1}^{n} \mu_j \rho_j^k + e(k), \qquad \mu_j \neq 0, \rho_j \in \mathbb{C}, \quad k = 0,1,\ldots,N$$

MP: given $\{\tau(k)\}$, find $\{\mu_j, \rho_j\}$ (and sometimes n)

- Q1: Well-posedness
  - Existense
  - Uniqueness
- Q2: Optimal performance tradeoffs (min-max / worst case)
- Q3: Algorithmic optimality

I) Uniqueness

#samples=#unknowns

$noise = 0$

(*) $\tau(k) = \sum_{j=1}^{n} \mu_j \rho_j^k$, $\mu_j \neq 0, \rho_j \in \mathbb{C},$ $k = 0,1, \dots, 2n-1$

<u>**Theorem 1: if the solution exists, it is unique up to a permutation.**</u>

Proof:

- Associate to each solution a point measure $\eta = \sum_{j=1,..,n} \mu_j \delta(t-\rho_j)$
- Suppose $\xi = \eta_1 - \eta_2 = \sum_{j=1,\dots,p} b_j \delta\left(t - w_j\right)$ with both η_1, η_2 solutions and $p \leq 2n$
- Then $\int z^k d\xi = 0,\ k = 0, \dots, 2n-1$
- This can be written as $\boldsymbol{V}_w \boldsymbol{b} = \boldsymbol{0}$ where V_w is the Vandermonde matrix
- Recall $|V_w| = \prod_{1 \leq i < j \leq p} (w_j - w_i)$ (try to prove!)
- Conclude that $\boldsymbol{b} = 0$

II) Existence

(*) $\tau(k) = \sum_{j=1}^{n} \mu_j \rho_j^k$, $\mu_j \neq 0, \rho_j \in \mathbb{C}$, $k = 0,1,\dots,2n-1$

- Turns out to be (somewhat) tricky
  - Consider: $\tau(0) = \cdots \tau(n-1) = 0$ but $\tau(k) \neq 0$ for some $n \leq k < 2n$

Definitions: $\widetilde{H}_n = \begin{bmatrix} \tau(0) & \cdots & \tau(n) \\ \vdots & \ddots & \vdots \\ \tau(n-1) & \cdots & \tau(2n-1) \end{bmatrix}$ $H_p = \begin{bmatrix} \tau(0) & \cdots & \tau(p-1) \\ \vdots & \ddots & \vdots \\ \tau(n-1) & \cdots & \tau(2p-2) \end{bmatrix}$

$|\boldsymbol{H_p}| \neq \mathbf{0}$ --> there is a unique solution **c** to $H_p\boldsymbol{c} = -\begin{bmatrix} \tau(p) \\ \vdots \\ \tau(2p-1) \end{bmatrix}$

<u>Theorem:</u> (*) has a (unique) solution $\boldsymbol{\eta}$ with $|\boldsymbol{supp\ \eta}| = \boldsymbol{p}$ if and only if

1. $\boldsymbol{p = rank(\widetilde{H}_n)}$
2. $|\boldsymbol{H_p}| \neq \mathbf{0}$
3. $\boldsymbol{Q_c(x) = x^p + \sum_{j=0}^{p-1} c_j x^j}$ **has no multiple roots.**

Ref: Y. I. Lyubich, "The Sylvester-Ramanujan System of Equations and The Complex Power Moment Problem," *The Ramanujan Journal*, vol. 8, no. 1, pp. 23–45, 2004.

Intermediate (generic) conclusions

(*) $$\tau(k) = \sum_{j=1}^{n} \mu_j \rho_j^k, \qquad \mu_j \neq 0, \rho_j \in \mathbb{C}, \qquad k = 0,1,\dots,2n-1$$

- "Unsolvability set" has Lebesgue measure 0 in $\mathbb{C}^{2n}$ (why?)
  - ➔ A generic perturbation will make the problem uniquely solvable
- Noiseless 1D SR square problem is generically solvable with infinite resolution
- For $N \geq 2n$ there is (generically) no solution

Extension #1

- Q: What if $\boldsymbol{Q_c(x)}$ has multiple roots? (later→ close by roots)
- A: change the model!

~~$$\tau(k) = \sum_{j=1}^{n} \mu_j \rho_j^k, \qquad \mu_j \neq 0, \rho_j \in \mathbb{C}, \qquad k = 0,1,\dots,2n-1$$~~

(**)

$$\tau(k) = \sum_{j=1}^{s} \sum_{\ell=0}^{d_j-1} \mu_{j,\ell} (k)_\ell \rho_j^{k-\ell}, \qquad \mu_{j,d_j} \neq 0, \rho_j \in \mathbb{C} \setminus \{0\},$$

$$\sum_j d_j \leq n, \quad (k)_\ell := k(k-1)\cdots(k-\ell+1), \quad k = 0,1,\dots,2n-1$$

<u>Theorem:</u> (**) has a (unique) solution $\boldsymbol{\eta = \sum_{j=1,\dots,s} \sum_{\ell=0,\dots,d_j-1} a_{j,\ell} \delta^{(\ell)}(t-\rho_j)}$ if and only if

1. $\boldsymbol{p = rank(\tilde{H}_n)}$
2. $\boldsymbol{|H_p| \neq 0}$
3. $\boldsymbol{Q_c(x) = x^p + \sum_{j=0}^{p-1} c_j x^j = \prod_{j=1,\dots,s} (x-\rho_j)^{d_j}}$.

Ref: D. Batenkov and Y. Yomdin, "Geometry and Singularities of the Prony mapping," *Journal of Singularities*, vol. 10, pp. 1--25, 2014, doi: 10.5427/jsing.2014.10a.

Univariate/1D (algebraic) extensions

$$\tau(k) = \sum_{j=1}^{n} \mu_j \rho_j^k = \left\langle \sum_{j=1}^{n} \mu_j \delta(\cdot - \rho_j), (\cdot)^k \right\rangle$$

$$\Rightarrow \sum_{\ell=0}^{n} c_\ell \tau(k+\ell) = 0, \qquad \forall k \in \mathbb{N}$$

- Modified Prony (Osborne & Smyth 1991) – linear recurrences

$$\sum_{j=1}^{m} c_j \left(\sum_{\ell=0}^{n} d_{\ell,j}(k)\, \Delta^\ell \right) \tau(k) = 0, \qquad \Delta \mathrm{q}(\mathrm{k}) = q(k+1) - q(k)$$

- Piecewise D-finite functions (Batenkov 2009)

$$(\mathfrak{D}f)(t) = \sum_{j,\ell} \mu_{j,\ell} \delta^{(\ell)}(t - t_j), \qquad \tau(k) = \langle f, (\cdot)^k \rangle, (\mathfrak{D}f)(t) = \sum p_j(t)(d \backslash \mathrm{dt})^j f(t)$$

  - Piecewise polynomials: $p_j(t) \equiv 1$

- Sparse sums of eigenfunctions (Peter&Plonka 2013, Stampfer&Plonka 2019)

$$f = \sum_{\lambda} c_\lambda v_\lambda, \qquad A v_\lambda = \lambda v_\lambda, \qquad \tau(k) = F(A^k f)$$

Extending recurrences - example

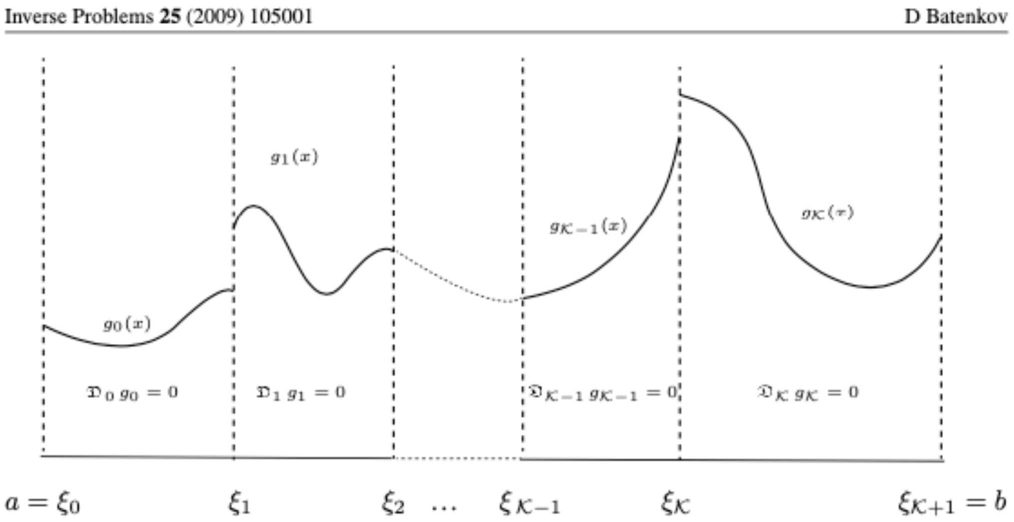


Figure 1. A piecewise *D*-finite function.

$$\mathfrak{D} = \sum_{j=0}^{N} p_j(x)\partial^j$$

$$p_j(x) = \sum_{i=0}^{k_j} a_{i,j} x^i, \quad a_{i,j} \in \mathbb{R}$$

$$m_k(g) = \int_a^b x^k g(x)\,\mathrm{d}x$$

Theorem: the moments $\{m_k\}$ satisfy a linear recurrence

$$\left(\prod_{n=0}^{K+1} (\mathrm{E} - \xi_n \mathrm{I})^N \Theta_{\mathrm{D}}(k, \mathrm{E})\right) m_k = 0$$

Proof: integration by parts

- Obtain Prony problem with RHS a lin.combination of moments
- Piecewise polynomials: $p_j(t) \equiv 1$
- Uniqueness: highly nontrivial

Research Q: can we replace moments with Fourier coefficients?

D. Batenkov, “Moment inversion problem for piecewise D-finite functions,” *Inverse Problems*, vol. 25, no. 10, p. 105001, Oct. 2009.

D. Batenkov and G. Binyamini, “Moment Vanishing of Piecewise Solutions of Linear ODEs,” in *Difference Equations, Discrete Dynamical Systems and Applications*, L. A. i Soler, J. M. Cushing, S. Elaydi, and A. A. Pinto, Eds. Springer Berlin Heidelberg, 2012, pp. 15–28.

A related extension

$$\hat{f}\,(k) = \sum_{j=1}^{K} \widehat{\varphi_j}(k) \exp(\imath k t_j)$$

Rational f-n (e.g. Lorenzian spectral shape)

Example: $$m_k := \left\{ \frac{c_1 \alpha^k + c_2 \beta^k}{1 + ak} \right\} \qquad \alpha := \exp(\imath t_1), \beta := \exp(\imath t_2)$$

Check: $$(1 + a(k+3))m_{k+3} - (\alpha + \beta)(1 + a(k+2))m_{k+2} + \alpha\beta(1 + a(k+1))m_{k+1} = 0$$

Peter & Plonka

$$f = \sum_{\lambda} c_\lambda v_\lambda, \qquad Av_\lambda = \lambda v_\lambda, \qquad \tau(k) = F(A^k f)$$

- $A: V \to V$ – a linear operator
- λ's = pairwise different eig's of A
- $F: V \to \mathbb{C}$ a known functional with $Fv_\lambda \neq 0$
- Exercise: derive the Prony problem

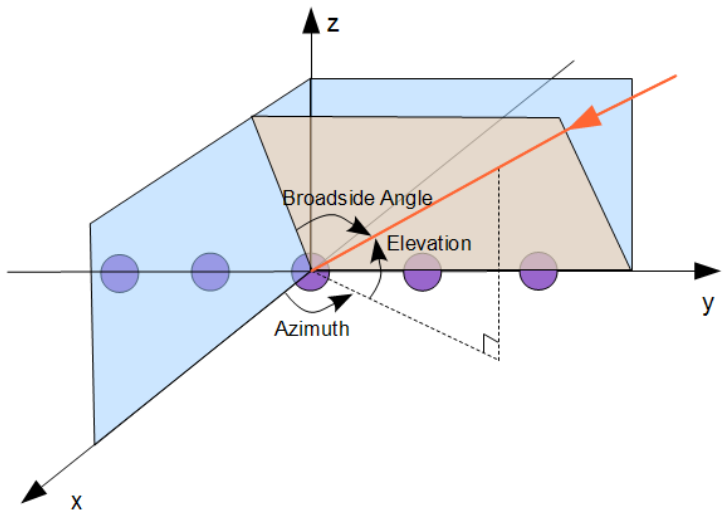
- Some recent extensions
- Some applications to numerical analysis and approximation theory

T. Peter and G. Plonka, "A generalized Prony method for reconstruction of sparse sums of eigenfunctions of linear operators," *Inverse Problems*, vol. 29, no. 2, p. 025001, Feb. 2013, doi: 10.1088/0266-5611/29/2/025001.

Multivariate Prony problem

$$\tau(\boldsymbol{x}) = \sum_{j=1}^{n} \mu_j \exp\left(i\langle \boldsymbol{t_j}, \boldsymbol{x}\rangle\right), \qquad \boldsymbol{x}, \boldsymbol{t_j} \in \mathbb{R}^d$$

$$H_{S_1,S_2} := [\tau(\boldsymbol{k}+\boldsymbol{l})]_{\boldsymbol{k}\in S_1}^{\boldsymbol{l}\in S_2} \qquad \boldsymbol{c} \in \ker \mathrm{H} \text{ iff } Q_{\boldsymbol{c}} \in I(\{\boldsymbol{t_j}\}) \text{ (for large enough } S_1, S_2\text{)}$$



- Projection methods (sampling on lines) – reduction to 1D (Diederichs, Cuyt, …)
- Connections to tensor decompositions (de Lathauwer et al. 2005, Comon&Usevich 2016)
- Kunis et al. (2015) – solution by finding common roots of multivariate polynomials

N-D: Unique reconstruction

Let

$$\Gamma_N = \{n \in \mathbb{N}_0^d \ : \ \prod (1 + n_j) \leq N\}$$

and for any $G \subset \mathbb{Z}^d$ define the Vandermonde vector

$$v_G(y) = (e^{2\pi i g \cdot y})_{g \in G}.$$

Lemma

If $|Y| \leq N$, then the vectors $(v_{\Gamma_N}(y),\ y \in Y)$ are linearly independent.

Lemma

There are no two different exponential sums with at most N summands that are equal on Γ_{2N}.

N-D: Unique reconstruction

Sampling set	Number of points	Reconstruction
Γ_{2N}	$\mathbf{O}(N \log^{d-1} N)$	no finite time
$\Gamma_{2^d N}$	$\mathbf{O}(N \log^{d-1} N)$	finite time, exponential complexity
$\Gamma_N + \lceil \Gamma_N \rceil$	$\mathbf{O}(N^2 \log^{2d-2} N)$	finite time, polynomial complexity

Table: Different Results

Open Question

Is for $\Gamma_{2^d N}$ exponential complexity necessary? I.e., is between $N\,\mathrm{polylog}(N)$ and $N^2\,\mathrm{polylog}(N)$ a conceptional barrier?
Also "open": How do the constants depend on d?

B. Diederichs, M. N. Kolountzakis, and E. Papageorgiou, "How many Fourier coefficients are needed?," *Monatsh Math*, Oct. 2022, doi: 10.1007/s00605-022-01792-0.

Recovery of algebraic N-D data

- Polygons P from complex moments (Golub et al. 1999)

$$\tau(k) = \iint_P z^k dxdy$$

- Quadrature domains Ω from moments (Gustaffson et al. 2000)

$$\tau(m,n) = \int_\Omega z^m \bar{z}^n dA(z), \qquad \int_\Omega f dA = u(f) \ \forall f \in L^1(\Omega) \cap \mathcal{A}(\Omega), \qquad u - \text{finite distr.}$$

- D-finite domains G (Batenkov et. al 2013)

$$\tau(m,n) = \iint_G x^m y^n dxdy$$

- Algebraic-exponential data (Lasserre&Putinar 2015)

$$\int_{g(\boldsymbol{x})<1} \boldsymbol{x}^{\boldsymbol{k}} \exp\big(p(\boldsymbol{x})\big)\, d\boldsymbol{x}, \qquad \boldsymbol{k} \in \mathbb{N}^d, \qquad \boldsymbol{x} \in \mathbb{R}^d$$

- ➢ Finite determinacy?
- ➢ Applications?

2D domains

- Let $P \subset \mathbb{C}$ be a polygon with vertices $z_1, \ldots, z_n$
- Measurements: $\mu_k(f) = \iint z^k f(x,y)\,\mathrm{d}x\,\mathrm{d}y$, $z = x + \imath y$ where $f = \chi_P$
- Turns out that there exist $c_1, \ldots, c_n \in \mathbb{C}$ s.t.

$$k(k-1)\mu_{k-2}(\chi_P) = \sum_{i=1}^{n} c_i z_i^k$$

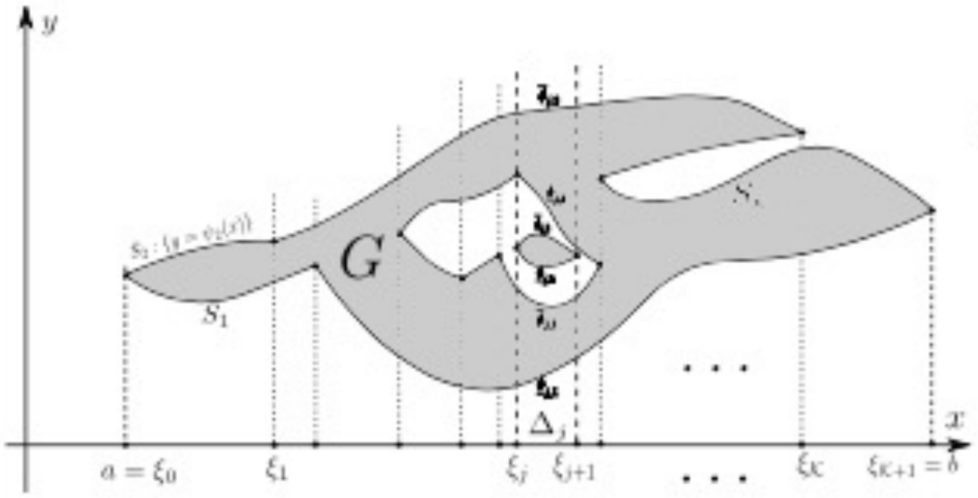
- Special case of *quadrature domains:* any analytic f (in particular $f(z) = z^k$) satisfies

$$\iint_\Omega f(x+\imath y)\,\mathrm{d}x\,\mathrm{d}y = \sum_{i=1}^{n}\sum_{j=0}^{k_j-1} c_{ij} f^{(j)}(z_i)$$

G. H. Golub, P. Milanfar, and J. Varah, “A stable numerical method for inverting shape from moments,” *SIAM Journal on Scientific Computing*, vol. 21, no. 4, pp. 1222–1243, 1999.
M. Elad, P. Milanfar, and G. H. Golub, “Shape from moments-an estimation theory perspective,” *Signal Processing, IEEE Transactions on*, vol. 52, no. 7, pp. 1814–1829, 2004.

B. Gustafsson, C. He, P. Milanfar, and M. Putinar, “Reconstructing planar domains from their moments,” *Inverse Problems*, vol. 16, no. 4, pp. 1053–1070, 2000.
B. Gustafsson, M. Putinar, E. B. Saff, and N. Stylianopoulos, “Bergman polynomials on an archipelago: Estimates, zeros and shape reconstruction,” *Advances in Mathematics*, vol. 222, no. 4, pp. 1405–1460, Nov. 2009, doi: 10.1016/j.aim.2009.06.010.

D-finite domains



$$m_{\alpha,\beta} = \int_a^b x^\alpha \Psi_\beta(x)\,\mathrm{d}x = \sum_{j=0}^{\mathscr{K}} \int_{\Delta_j} x^\alpha \Psi_{\beta,j}(x)\,\mathrm{d}x$$

$$\Psi_{\beta,j} = \int_{\underline{\phi}_{j,1}(x)}^{\overline{\phi}_{j,s_j}(x)} y^\beta \chi_G(x,y)\,\mathrm{d}y$$

$$= \frac{1}{\beta+1} \sum_{\ell=1}^{s_j} \left\{ \overline{\phi}_{j,\ell}^{\beta+1}(x) - \underline{\phi}_{j,\ell}^{\beta+1}(x) \right\}$$

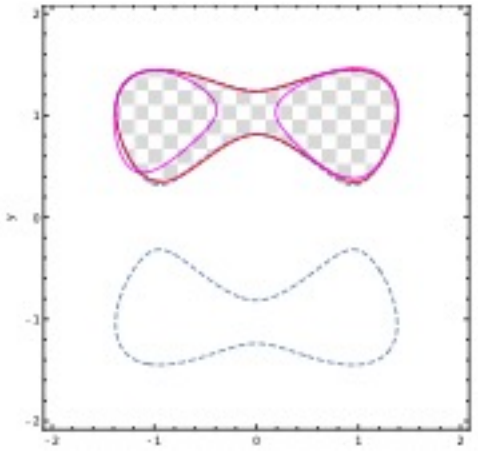
- Ψ_β are piecewise D-finite, are reconstructed via the 1D algorithm (need to use desingularization).
- $\{\phi_{j,\ell}\}$ are reconstructed pointwise via solving Prony-type system.
- Explicit formulas and estimates for $\{\phi_{j,\ell}\}$ – polynomials.

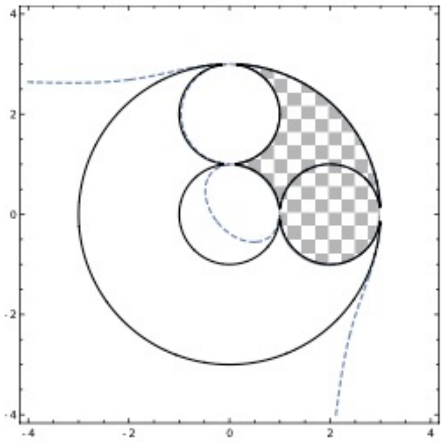
D. Batenkov, V. Golubyatnikov, and Y. Yomdin, “Reconstruction of planar domains from partial integral measurements,” in *Complex analysis and dynamical systems V*, vol. 591, Amer. Math. Soc., Providence, RI, 2013, pp. 51–66.

Algebraic-exponential data

$$m_\alpha = \int_G x^\alpha \mathrm{d}\mu_f(x), \quad \alpha \in \mathbf{N}^n$$

, for example with $f(x) = \exp(p(x))$





$$\int_{g(\boldsymbol{x})<1} \boldsymbol{x}^{\boldsymbol{k}} \exp\big(p(\boldsymbol{x})\big)\, d\boldsymbol{x}, \qquad \boldsymbol{k} \in \mathbb{N}^d, \qquad \boldsymbol{x} \in \mathbb{R}^d$$

Inverse problem: recover p and g

Fact: there is a certain "annihilation" $\widehat{\mathbf{M}}_k^d(\mathbf{y}) \begin{bmatrix} -1 \\ \mathbf{g} \end{bmatrix} = 0.$

J. B. Lasserre and M. Putinar, "Algebraic–exponential Data Recovery from Moments," *Discrete & Computational Geometry*, vol. 54, no. 4, pp. 993–1012, Dec. 2015
F. Bréhard, M. Joldes, and J.-B. Lasserre, "On Moment Problems with Holonomic Functions," in *ISSAC 2019*

Conclusions

- Algebraic models are powerful tools for representation of geometric data
- Research topic of current interest
- Numerical stability & resolution – TBD