Lecture 8

Super-Resolution and ℓ_1 minimization

Topics in Inverse Problems
Fall 2022

Recap - sparsity and ℓ_1

- ℓ_0"norm": $\|x\|_0 := \{\# i : x_i \neq 0\}$
- "Sparse prior": $\|x\|_0$ is small
- Sparse regularization: $\arg\min_x \|x\|_0$ s.t. $\|Ax - y^\delta\|_2 \leq \epsilon$
- Convex relaxation:
    - $\arg\min_x \|x\|_1$ s.t. $\|Ax - y^\delta\|_2 \leq \epsilon$
    - $\arg\min_x \|Ax - y^\delta\|_2 + \lambda \|x\|_1$
    - Today will also see: $\arg\min_x \|Ax - y^\delta\|_1$ s.t. $x \geq 0$

Super-resolution and sparsity

- $x = \sum_j a_j \delta_{t_j} \in \mathbb{C}^N$
- $\operatorname{supp}(x) = T$ (unknown)
- For today, assume that $\forall j : t_j \in \left\{0, \frac{1}{N}, \ldots, \frac{N-1}{N}\right\}$ (compare: $t_j \in [0,1)$)
- Low-frequency measurements:
  - Discrete: $\mathbb{C}^n \ni y = F_n x + w$,
    $F_n = \{$first $n \ll N$ rows of $N \times N$ DFT matrix$\} \in \mathbb{C}^{N \times n}$
  - Continuous: $y(\omega) = \sum_j a_j \exp(2\pi\imath\omega t_j) + e(\omega), \ |\omega| \leq \Omega \ll N$

Upper bound on the error

- Notation: $F_{n,T} = T-$column slice of F_n
- $\sigma_{\min,T}(F_n) = \sigma_{\min}(F_{n,T})$
- For $a \in \mathbb{N}$, $\sigma_{\min,a} := \min_{|T|=a} \sigma_{\min,T}$
- Suppose $\|x\|_0 \leq s$ and $\|y - F_n x\|_2 \leq \epsilon$
- Claim: any other $\|x_1\|_0 \leq s$, $\|Ax_1 - y\|_2 \leq \epsilon$ will satisfy
  $\|x - x_1\|_2 \leq \dfrac{2\epsilon}{\sigma_{\min,2s}(F_n)}$

Proof

$$\|x - x_1\|_0 \le 2s$$

$$\left.\begin{aligned} \|y - F_n x\|_2 &\le \varepsilon \\ \|y - F_n x_1\|_2 &\le \varepsilon \end{aligned}\right\} \Rightarrow \|F_n(x - x_1)\|_2 \le 2\varepsilon$$

$$2\varepsilon \ge \|F_n(x - x_1)\|_2 \ge \sigma_{min,2s}(F_n)\|x_1 - x\|_2$$

- When $\sigma_{min,2s} > 0$
- Estimate from below.

Matching lower bound

Exercise: given $\epsilon > 0$, there exist $\tilde{x}_1, \tilde{x}_2$ (not both zero) with $\|\tilde{x}_1\|_0, \|\tilde{x}_2\|_0 \leq s$ such that

1. $\|F_n(\tilde{x}_1 - \tilde{x}_2)\|_2 \leq \epsilon$;
2. $\|\tilde{x}_1 - \tilde{x}_2\|_2 = \dfrac{\epsilon}{2\sigma_{\min,2s}(F_n)}$

What is $\sigma_{\min,2s}(F_n)$?

- Can compute explicitly:
$$\frac{1}{N} F_n^* F_n \asymp \frac{1}{W} \cdot \left[\frac{\sin(W(i-j))}{(i-j)} \right]_{i,j=0,\ldots,N-1}, \quad W := \frac{n}{N}, \quad n, N \to \infty$$

- $$\mathrm{sinc}(Wt) := \begin{cases} \frac{\sin(Wt)}{Wt} & t \neq 0 \\ 1 & t = 0 \end{cases} = \frac{1}{2} \int_{-W}^{W} \exp(\imath \omega t) d\omega = \lim_{N \to \infty} \frac{1}{2N} \sum_{k=-N}^{N} \exp\left(\imath \frac{k}{N} \Omega t \right)$$

- ==> Can apply “Bell Labs theory”

- Theorem: if $T = \left\{ 0, \frac{1}{N}, \ldots, \frac{s-1}{N} \right\}$ then $\sigma_{\min,T}(F_n) \asymp_s \left(\frac{n}{N} \right)^{s-1}$

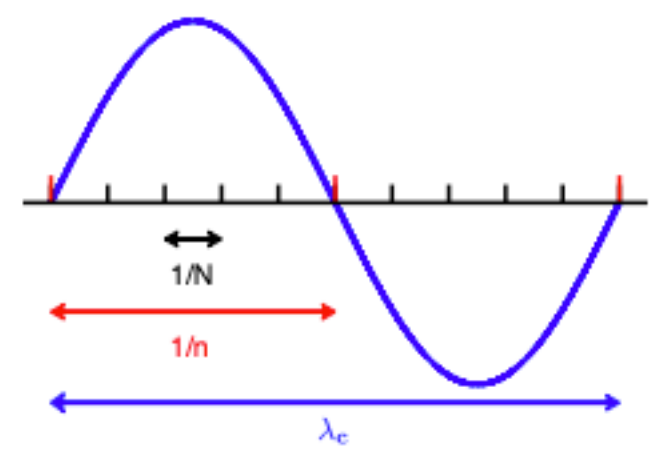
SR via sparsity

- Error $\asymp \left(\frac{N}{n}\right)^{2s-1} \epsilon$
- Too bad if s is reasonably high
- Will show how to overcome this by imposing further constraints on T

SR with separation

$$x^* = \arg\min_x \|x\|_1 \ \textbf{s.t.}\ \Big\|\underbrace{\frac{1}{N}F_n^* y^\delta}_{:=u} - P_n x\Big\|_2 \le \delta, \quad P_n := \frac{1}{N}F_n^* F_n \in \mathbb{C}^{N\times N}$$

- Condition: restrict $\|t_i - t_j\|_{\mathbb{T}} \gtrapprox 1/n$
- Theorem: $\|x^* - x\|_1 \lessapprox \left(\frac{N}{n}\right)^2 \delta$



E. J. Candès and C. Fernandez-Granda, "Towards a Mathematical Theory of Super-resolution," *Comm. Pure Appl. Math*, vol. 67, no. 6, pp. 906–956, Jun. 2014.

Proof #2: $n = 2f_c + 1$ $\lambda_c = \frac{1}{f_c}$

„Hard analysis" ("existence of a dual certificate")

Proposition: Suppose $\|t_i - t_j\|_{\mathbb{T}} \geq \frac{2}{f_c} \approx \frac{4}{n}$

For any $|\delta_j| = 1$, $j = 1, \dots, |T|$,

there exists a low frequency trigonometric polynomial

$$q(t) = \sum_{k=-f_c}^{f_c} c_k e^{2\pi i k t} \quad \text{s.t.}$$

$$\begin{cases} q(t_j) = \delta_j, & t_j \in T \\ |q(t)| < 1, & t \in [0,1) \setminus T \end{cases}$$

The construction

$$q(t) = \sum_{t_j \in T} \alpha_j K\left(t - t_j\right) + \beta_j K'\left(t - t_j\right)$$

$$q\left(t_k\right) = \sum_{t_j \in T} \alpha_j K\left(t_k - t_j\right) + \beta_j K'\left(t_k - t_j\right) = v_k \quad \forall t_k \in T$$

$$q'\left(t_k\right) = \sum_{t_j \in T} \alpha_j K'\left(t_k - t_j\right) + \beta_j K''\left(t_k - t_j\right) = 0 \quad \forall t_k \in T$$

$$K(t) = \left[\frac{\sin\left(\left(f_c/2 + 1\right)\pi t\right)}{\left(f_c/2 + 1\right)\sin(\pi t)}\right]^4, \quad 0 < t < 1$$

"Robust" Null space property" [key in CS/sparse recovery etc.]

Lemma: Suppose $P_n h = 0$

Denote $h_T \equiv h|_T \in \mathbb{C}^N$

$h_{T^c} \equiv h|_{T^c}$

$\exists \rho$: $0 < \rho < 1$ s.t. (not depending on h).

$$\|h_T\|_1 \le \rho \|h_{T^c}\|_1$$

Pf: Let $h_T(j) = |h_T(j)| e^{i\phi_j}$, $\sigma_j = e^{-i\phi_j}$

lets take f from Prop.

(dual cert.) $r_j = f(j/N) \Longrightarrow r = P_n r$ (check).

By construction, $\exists \rho$ s.t

$\forall_j \; j/N \notin T \quad |r_j| \leq \rho < 1$

- $\langle r, h \rangle = \langle P_n r, h \rangle = \langle r, P_n h \rangle = 0$

- $0 = \langle r, h \rangle = \langle r_T, h_T \rangle + \langle r_{T^c}, h_{T^c} \rangle \geq \|h_T\|_1 - \|r_{T^c}\|_\infty \|h_{T^c}\|_1$

$\sum_{j \in T} |h_T(j)| = \|h_T\|_1$

$\geq \|h_T\|_1 - \rho \cdot \|h_{T^c}\|_1$

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$$x^* = \arg\min_x \|x\|_1 \textbf{ s.t. } \Big\| \underbrace{\frac{1}{N} F_n^* y^\delta}_{:=u} - P_n x \Big\|_2 \le \delta, \quad P_n := \frac{1}{N} F_n^* F_n \in \mathbb{C}^{N \times N}$$

Theorem: $\|x^* - x\|_1 \lessapprox \left(\frac{N}{n}\right)^2 \delta$

Pf of Thm:

1) $h = x^* - x = \underbrace{P_h h}_{h_L} + \underbrace{(I - P_n) h}_{h_H}$

notice $P_n h_H = 0 \Rightarrow \|(h_H)_T\|_1 \le \rho \|(h_H)_{T^c}\|_1$ (*)

N.S.P

$$\|(h_H)_T\|_1 \leq \rho \|(h_H)_{T^c}\|_1 \quad (*)$$

2) $\|x\|_1 \geq \|x^*\|_1 = \|x+h\|_1 = \|x+h_L+h_H\| \geq \|x+h_H\|_1 - \|h_L\|_1$

x^* is a solution

$$\geq \|x\|_1 - \|(h_H)_T\|_1 + \|(h_H)_{T^c}\|_1 - \|h_L\|_1$$

Side comp

$x + h_H = x + (h_H)_T + (h_H)_{T^c}$

$\|x+h_H\|_1 = \|x+(h_H)_T\|_1 + \|(h_H)_{T^c}\|_1$

$$\underset{(*)}{\geq} \|x\|_1 + (1-\rho)\|(h_H)_{T^c}\|_1 - \|h_L\|_1$$

$$\|h_L\|_1 \geq (1-\rho)\|(h_H)_{T^c}\|_1 \quad (**)$$

$$\|h_H\|_1 = \|(h_H)_T\|_1 + \|(h_H)_{T^c}\|_1 \underset{(*)}{\leq} (1+\rho)\|(h_H)_{T^c}\|_1 \underset{(**)}{\leq} \frac{1+\rho}{1-\rho}\|h_L\|_1$$

$$\|h_H\|_1 \le \frac{1+\rho}{1-\rho} \| h_L \|_1 \quad (**)$$

$$\|h\|_1 \le \|h_H\|_1 + \|h_L\|_1 \le \frac{2}{1-\rho}\|h_L\|_1 = \frac{2}{1-\rho} \|P_h h\|_1 \le \frac{4\delta}{1-\rho}$$

$P_h h = P_h x^* - P_h x$

$\|P_h x^* - u\| \le \delta$

$\|P_h x - u\| \le \delta$

Turns out that

$$|\phi(j/N)| \lesssim 1 - 0.0883\left(\frac{h}{N}\right)^2 = \rho$$

$j/N \in T$ [Technical proof]

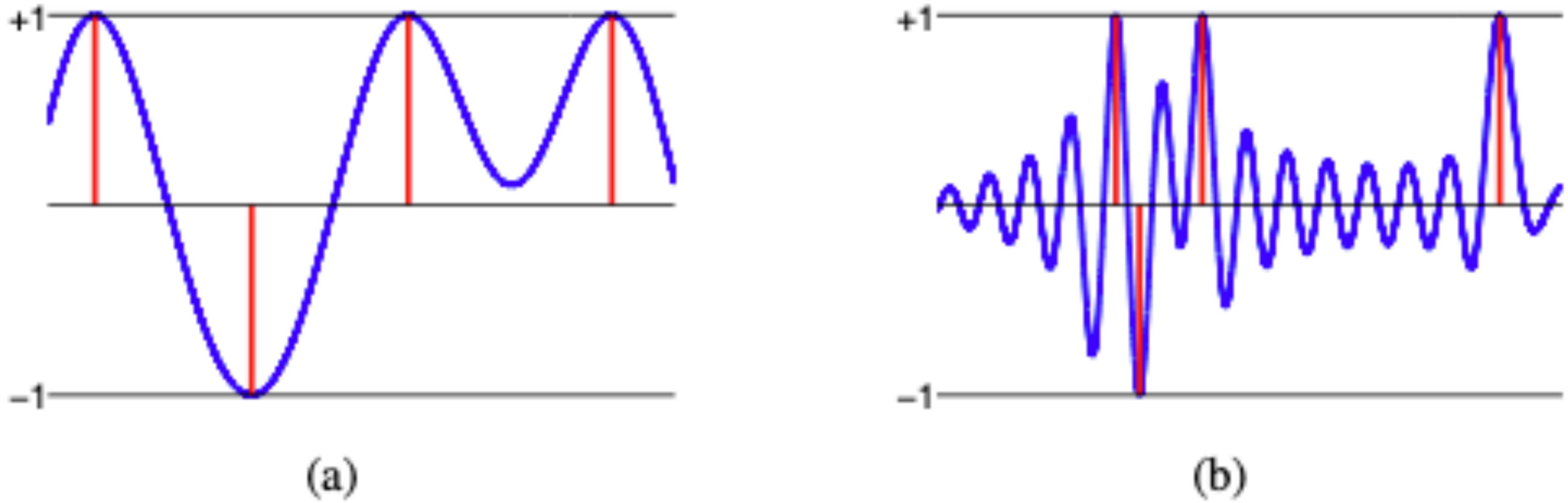

FIGURE 2.1. (a) Low-frequency polynomial interpolating a sign pattern in which the support is well separated and obeying the off-support condition (2.2). In (b), we see that if the spikes become very near, we would need a rapidly (high-frequency) interpolating polynomial in order to achieve (2.2). This is the reason that there must be a minimum separation between consecutive spikes.

Further remarks

- Trade separation for stability and tractability
  - separation constraint is essential for convex methods to work
- "Super-localization": exact solution for $\delta = 0$
- Started a huge (but not \HUGE like CS) field
  - Continuous setting
  - Other measurements kernels

Towards a mathematical theory of super-resolution

…, C Fernandez-Granda - Communications on pure …, 2014 - Wiley Online Library

This paper develops a mathematical theory of super-resolution. Broadly speaking, super-resolution is the problem of recovering the fine details of an object—the high end of its spectrum—from coarse scale information only—from samples at the low end of the …

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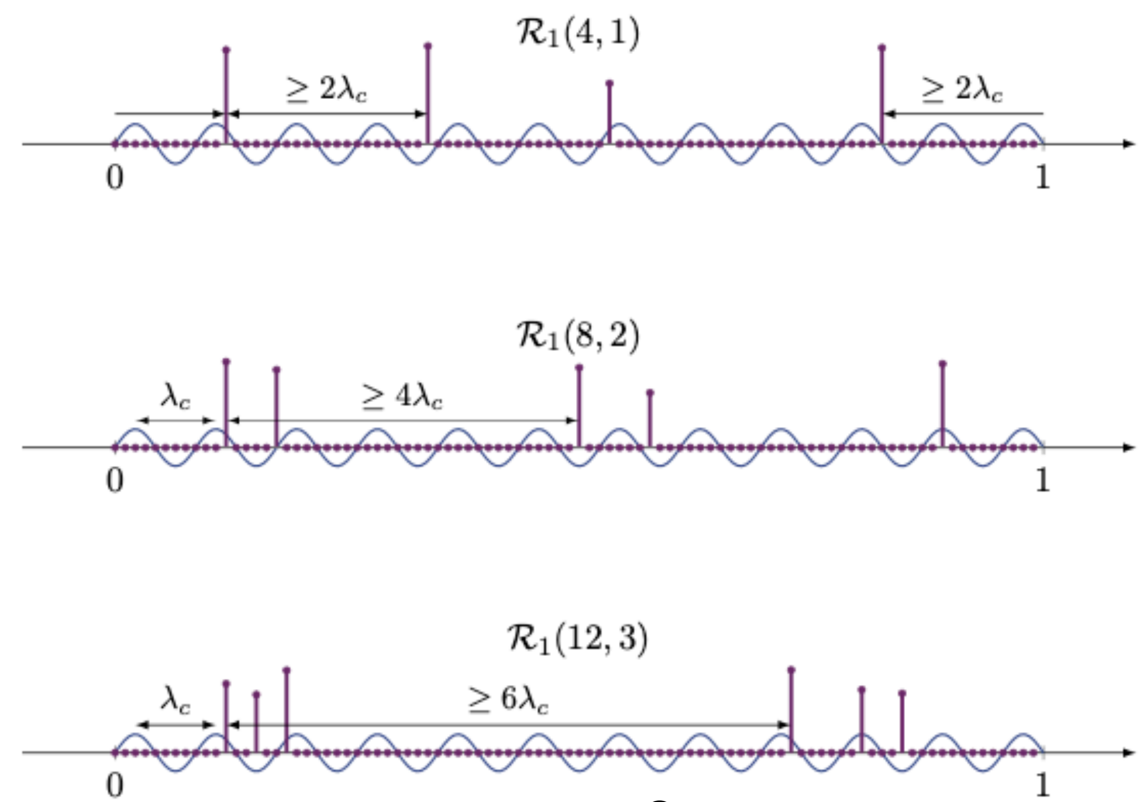
SR with positivity

- Positive sources: relevant to e.g. microscopy, astronomy
- We don't need separation anymore
- Instead, assume "clustered sparsity" or "Rayleigh-regularity"



$$\lambda_c = \frac{2}{n}$$

<u>Definition:</u> $\mathscr{R}^+(d, r)$ **is the set of signals** x **s.t.** $T = \text{supp}(x) = T_1 \cup \cdots \cup T_r$ **with** $\{T_i\}$ **disjoint, and** $\forall i : \left| I \cap T_i \right| \leq 1$ **for every interval** I **of length** $\frac{d}{n}$

$$x^* := \arg\min_x \|u - P_n x\|_1 \text{ s.t. } x \geq 0.$$

<u>Theorem:</u> if $\|u - P_n x\|_1 \leq \delta$ **and** $x \in \mathscr{R}^+(3.74r, r)$ **then** $\|x^* - x\|_1 \lessapprox \left(\frac{N}{n}\right)^{2r} \delta$

V. I. Morgenshtern and E. J. Candès, "Super-Resolution of Positive Sources: The Discrete Setup," *SIAM Journal on Imaging Sciences*, vol. 9, no. 1, pp. 412–444, Jan. 2016, doi: 10.1137/15M1016552.

Proof #3 As before, $h = x^* - x$, $x^* \geq 0$, $x \geq 0$

Notice: h can be negative only on T.

Let $S = \left\{ j/N : h(j) < 0 \right\} \subset T$.

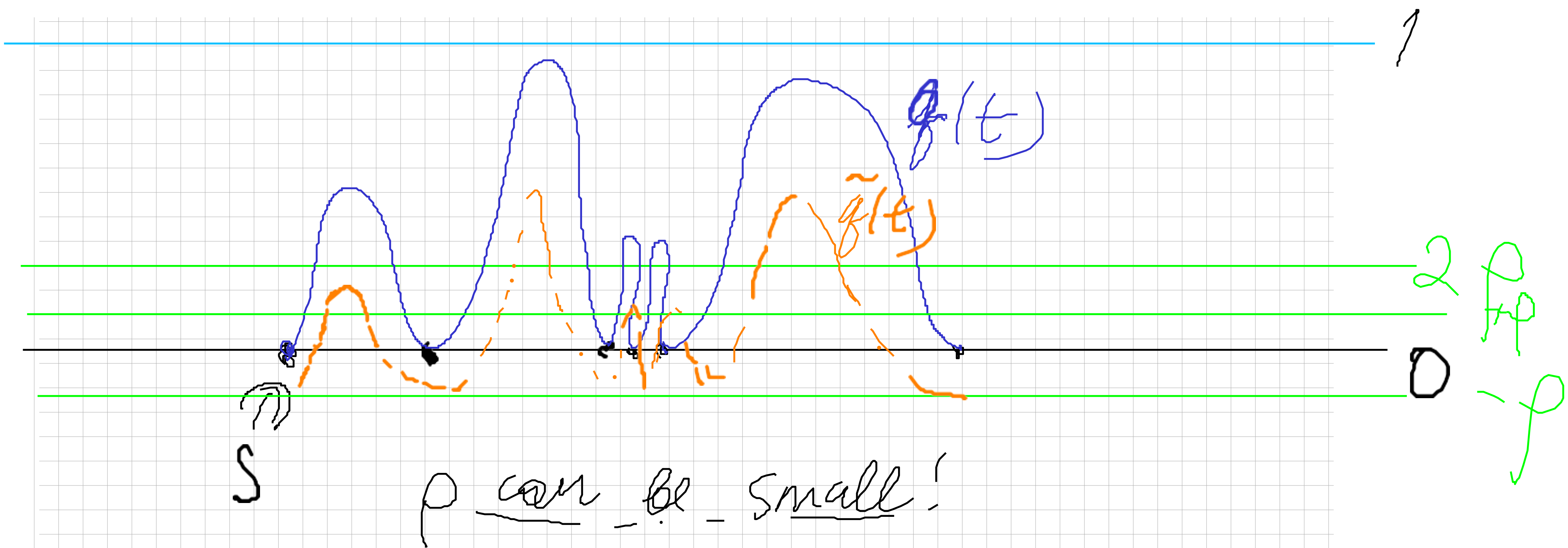
Lemma: $\exists\, 0 < \rho < 1/2$ and a low-freq. trig. polynomial

$\left(f(t) = \sum_{k=-f_c}^{f_c} c_k e^{2\pi i k t} \right)$ s.t. $\|f\|_\infty \leq 1$,

$r_j := f(j/N)$

$P_h \cdot r = r$

$$\begin{cases} r_j = 0 & j/N \in S \\ r_j > 2\rho & j/N \notin S \end{cases}$$



Technical pf: $\rho \gtrsim \left(\frac{h}{N}\right)^{2r}$

define $\tilde{r} := r - \rho$ (new D.C.) $\tilde{q}(t) = q(t) - \rho$

- $\|\tilde{r}\|_\infty \le 1 - \rho$ - $|\tilde{r}_j| \ge \rho \quad \forall j$

- $\operatorname{sign} h_j = \operatorname{sign} \tilde{r}_j \quad \forall j$!

New Dual Certificate

- $\|\tilde{r}\|_\infty \leq 1-\rho$ - $|\tilde{r}_j| \geq \rho \quad \forall j$
- $\operatorname{sign} h_j = \operatorname{sign} \tilde{r}_j \quad \forall j$!

Now: $|\langle \tilde{r}, h\rangle| = |\langle P_h \tilde{r}, h\rangle| = |\langle \tilde{r}, P_h h\rangle| \leq \|\tilde{r}\|_\infty \|P_h h\|_1 \leq 2(1-\rho)\delta$

$P_h h = P_h x^* - P_h x \qquad \|P_h h\|_1 \leq 2\delta$

$\|u - P_h x\|_1 \leq \delta$

$\|u - P_h x^*\|_1 \leq \|u - P_h x\|_1 \leq \delta$

$|\langle \tilde{r}, h\rangle| = |\sum \tilde{r}_j h_j| = \sum |\tilde{r}_j||h_j| \geq \rho \|h\|_1$

$$\Rightarrow \boxed{\|h\|_1 \leq \frac{2(1-\rho)}{\rho}\delta}$$