Lecture 6

Introduction to super-resolution

Topics in Inverse Problems
Fall 2021

Nyquist sampling theorem

Suppose $f \in PW_{\Omega}$**, then, for** $T < \dfrac{\pi}{\Omega}$

$$f(x) = \sum_{n=-\infty}^{+\infty} f(nT)\,\mathrm{sinc}\,\frac{(x-nT)}{T}$$

- **So, a function in** PW_{Ω} **is uniquely determined by its samples on** $\left\{ n\dfrac{\pi}{\Omega} \right\}_{n\in\mathbb{Z}}$

Nyquist criterion: Resolution $\dfrac{\pi}{\Omega} \Longleftrightarrow$ **band limit** Ω

Resolution limit

$$\int K(t,s)f(s)ds = g(t)$$

- Given an output of a band-limited system $Ax^\delta = y^\delta$, what is the “resolution” of x^δ?

- How does it depend (if at all) on δ?

Answer 1: Highest frequency

- With analogy to Nyquist, we would expect that $Resolution \approx \{Highest\ \ \textbf{frequency}\ \ in\ \ x^{\delta}\}^{-1}$
- Frequency = index of singular value (say, N)
- N is morally equivalent to “# degrees of freedom”

Analysis via SVD

- Recall: $x^\delta = \sum_{i=1}^{N} \frac{1}{\mu_i} \langle y^\delta, \psi_i \rangle \varphi_i$
- Error bound: $\|\delta x\| \leq \frac{\delta}{\alpha} + \left(\sum_{\mu_i < \alpha} \left| \langle x, \varphi_i \rangle \right|^2 \right)^{1/2}$
- So, choose N with $\mu_N \approx \frac{\delta}{\left| \langle x, \varphi_N \rangle \right|}$

For example:

$$\mu_n \sim e^{-\beta n}, \langle x, \varphi_n \rangle \sim n^{-\gamma}$$

$$N \sim \frac{1}{\beta} \log \frac{1}{\delta}$$

$$\|\delta x\| \sim |\log \delta|^{-\gamma}$$

What is the rate of decay of μ_n?

Decay of singular values

- Can (should) look for asymptotic formulas
    - For compact operators, largely depends on the smoothness of the kernel
    - Exponential decay = “severely ill-posed”
    - Polynomial decay = “mildly ill-posed”
- Sometimes, very accurate asymptotics and distributional results are known

Exercise: derivative operator

- Consider $\int_0^1 K(s,t)f(t)dt = g(s), 0 \le s \le 1$ with $K(s,t) = \begin{cases} s(t-1), & s < t \\ t(s-1), & s \ge t \end{cases}$
- Solution: $f(t) = g''(t),\ 0 \le t \le 1$. Check via differentiation under the integral
- But also: $K(s,t) = -\dfrac{2}{\pi^2} \sum_{k=1}^{\infty} \dfrac{\sin(k\pi s)\sin(k\pi t)}{k^2}$ (look up: Mercer's theorem)
- Show that

$$\mu_k = \frac{1}{(k\pi)^2}, \quad \varphi_k(s) = \sqrt{2}\sin(k\pi s), \quad \psi_k(t) = -\sqrt{2}\sin(k\pi t), \quad k = 1,2,\ldots$$

Weyl-Courant minimax principle

Let $A : X \to X$ **compact, self-adjoint,** $\langle Ax, x \rangle \geq 0$ **(nonnegative), with eigenvalues** $\lambda_1 \geq \lambda_2 \geq \ldots$ **Then** $\lambda_{n+1} = \inf_{L \subset X, \dim(L)=n} \sup \left\{ (A\varphi, \varphi) : \varphi \in L^{\perp}, \, \|\varphi\| = 1 \right\}$**.**

Proof**:** see e.g. Kress, Theorem 15.14.

What about singular values?

Theorem 15.17. *Let $A, B : X \to Y$ be compact linear operators. Then for the nonincreasing sequence of singular values there holds*

$$\mu_1(A) = \|A\| = \sup\{\|A\varphi\| : \|\varphi\| = 1\} \tag{15.21}$$

and

$$\mu_{n+1}(A) = \inf_{\psi_1,\dots,\psi_n \in X} \sup\{\|A\varphi\| : \varphi \perp \psi_1, \dots, \psi_n,\ \|\varphi\| = 1\} \tag{15.22}$$

for all $n \in \mathbb{N}$ and

$$\mu_{n+m+1}(A + B) \le \mu_{n+1}(A) + \mu_{m+1}(B) \tag{15.23}$$

for all $n, m = 0, 1, 2, \dots$.

(Kress, p.233)

Analytic kernels

If $K(x,t) = K(t,x) \in C[-1,1]^2$, and for each $t \in [-1,1]$ there is an analytic continuation to $K(z,t)$ for z inside E_R, which is uniformly bounded in z, t in this range, and if the operator

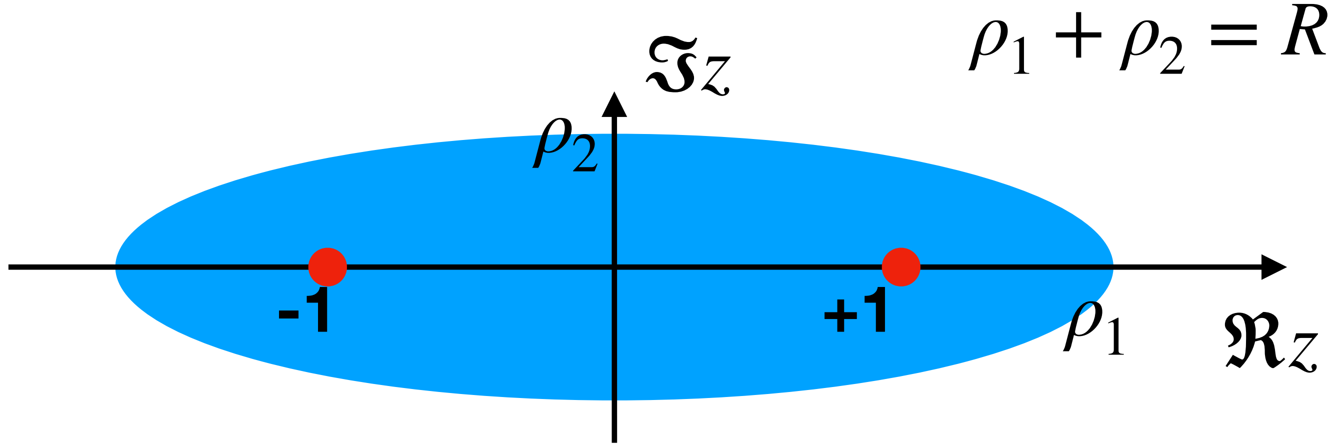
$$Tf(x) = \int_{-1}^{1} K(x,t) f(t) dt$$

has eigenvalues

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| \geq \cdots$$

then $\lambda_n = O\left(R^{-n}\right)$

Bernstein ellipse E_R



G. Little and J. B. Reade, "Eigenvalues of Analytic Kernels," *SIAM J. Math. Anal.*, 15(1), pp. 133–136, 1984

Proof outline

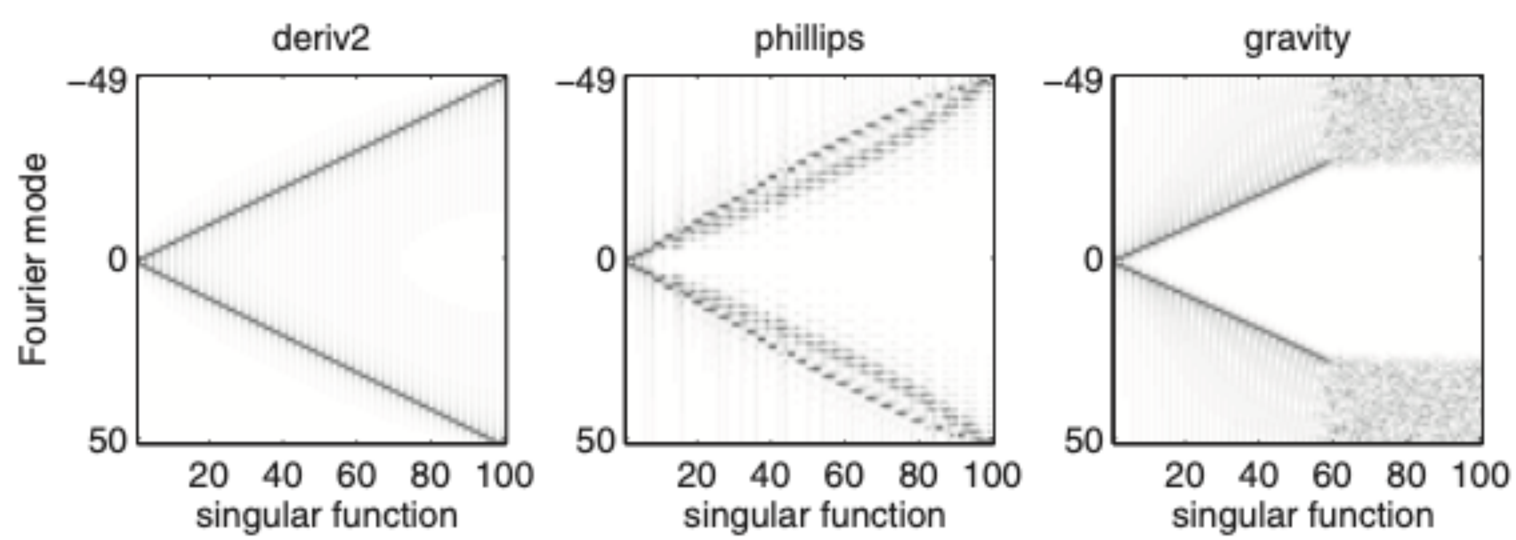
- $T_n(x)$ - Chebyshev polynomials (*) (orthogonal wrt scalar product $\langle f, g \rangle = \int_{-1}^{1} \frac{f(t)g(t)}{\sqrt{1-t^2}} dt$)
- Fact: if f is analytic inside E_R and continuous on the boundary, then $f(z) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n T_n(z), \quad z \in E_R$, and $a_n = O(R^{-n})$.
- Write $K(x,t) = \frac{1}{2}a_0(t) + \sum_{n=1}^{\infty} a_n(t) T_n(x)$, with a_n continuous
- Approximate K by $S_n(x,t) = \frac{1}{2}a_0(t) + \sum_{k=1}^{n} a_k(t) T_k(x)$ - finite rank operator of rank $\leq n+1$.
- $\| S_n - K \|_{\infty} = O\left(R^{-n}\right)$
- Conclude: $\mu_{n+2}(A_K) \leq \mu_1(A_K - A_{S_n}) + \mu_{n+2}(A_{S_n}) = O(R^{-n})$

Answer 2: oscillations of the solution

φ_n are usually oscillatory, with #oscillations growing with n - so a reasonable measure of "resolution" as well (although more of an "average")

Example: $Af = \int_{-\pi}^{\pi} K(s,t)f(s)ds, \quad K(\pi,t) = K(-\pi,t)$ **a.e.** x (Hansen Section 2.5)

Estimate $\left| \left\langle \varphi_j, e^{iks} \right\rangle \right|$



Small singular values ($j \gg 1$) correspond to high frequencies

Total positivity and Chebyshev systems

If every φ_n has at most n zeros, what about linear combinations?

$\{\varphi_j\}_{j=1}^n$ **is a <u>Chebyshev system</u> if** $\det \varphi_j(x_i) > 0$ **for any** $x_1 < \ldots < x_n$

If $\{\varphi_j\}_{j=1}^n$ **is a CS then** $\sum_{i=1}^n a_i \varphi_i$ **has at most** $n-1$ **zero crossings.**

K **is <u>totally positive</u> if** $\det K(x_i, y_j) \geq 0$ **for any** $x_1 < \ldots < x_n, y_1 < \ldots < y_n$

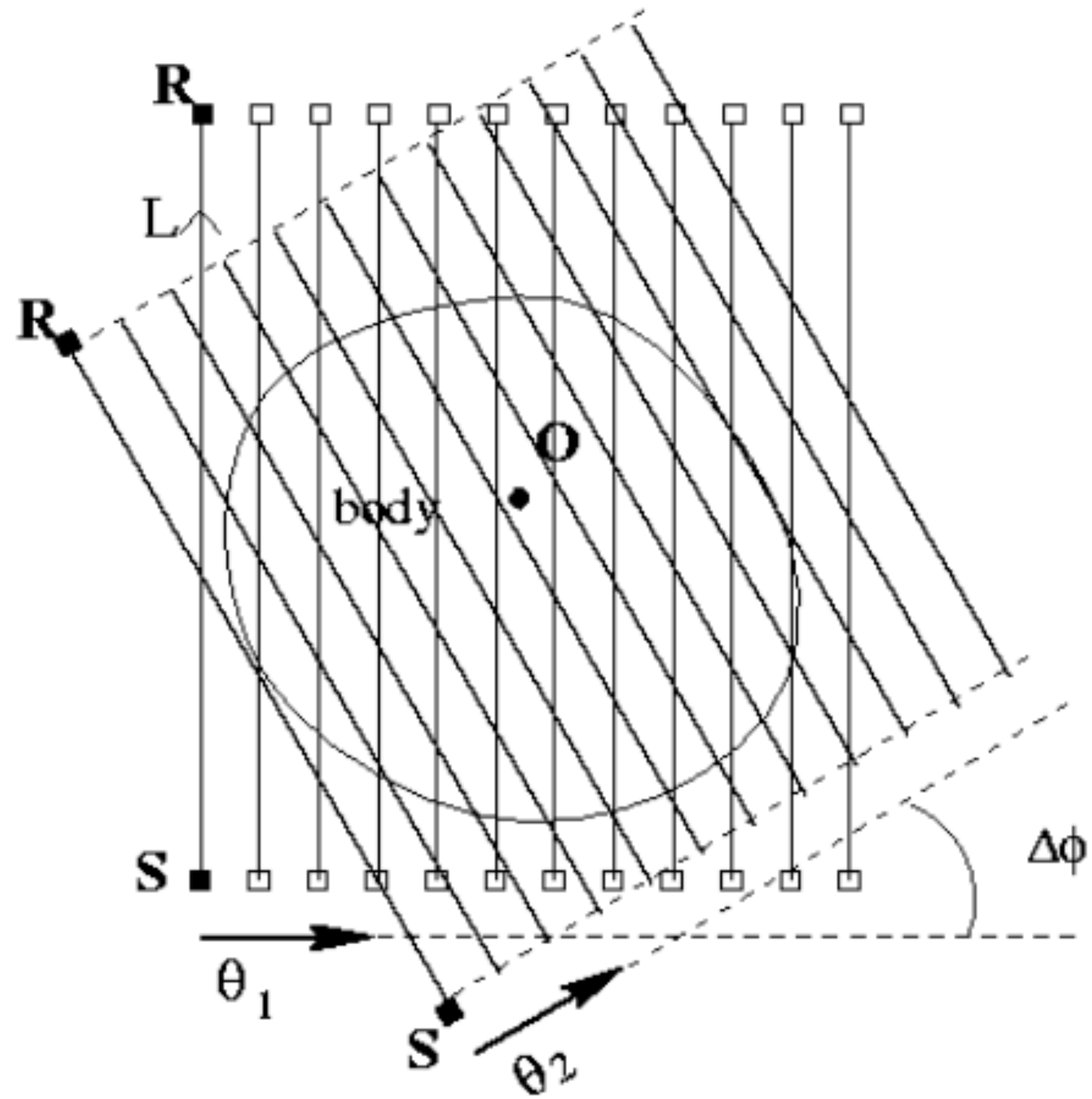
If K **is strictly totally positive then** $\{\varphi_i\}_{i=1}^n$ **is a CS (for every** $n \geq 1$**)**

Example: $K(x, y) = e^{xy}$

S. Karlin. 1968. Total Positivity, Vol. 1. Stanford University Press.
"Total positivity and its applications" M. Gasca (ed.) C.A. Micchelli (ed.) , Kluwer Acad. Publ. (1996)

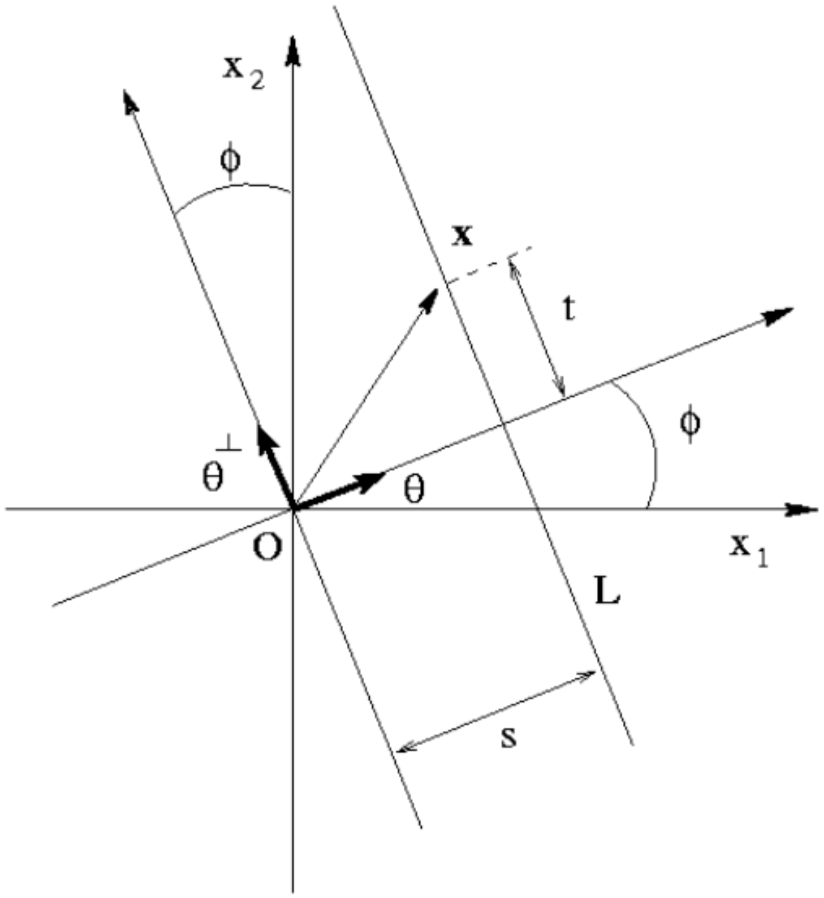
Computed tomography (CT)

Ref: Bertero & Bocacci, chapters 8,11



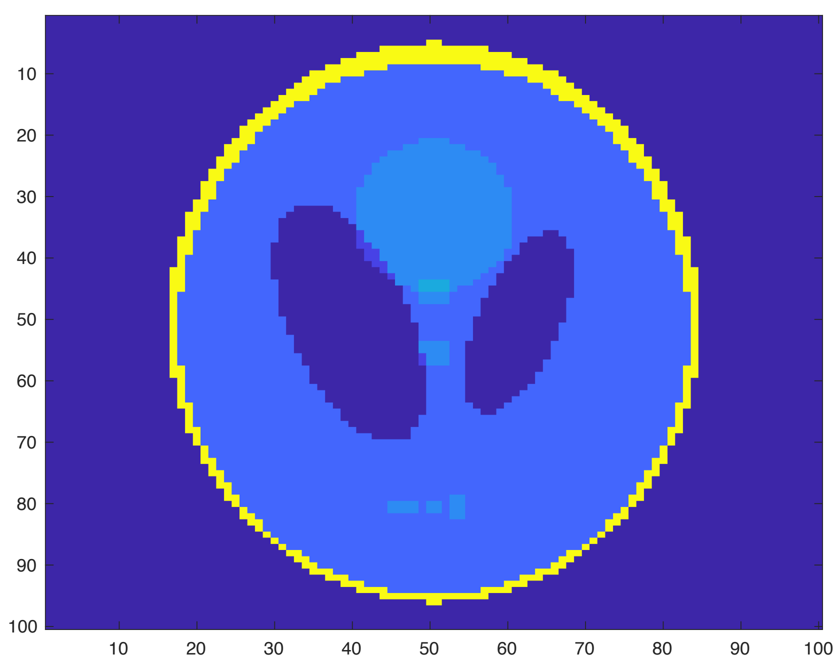
Task: reconstruct an object from its projections

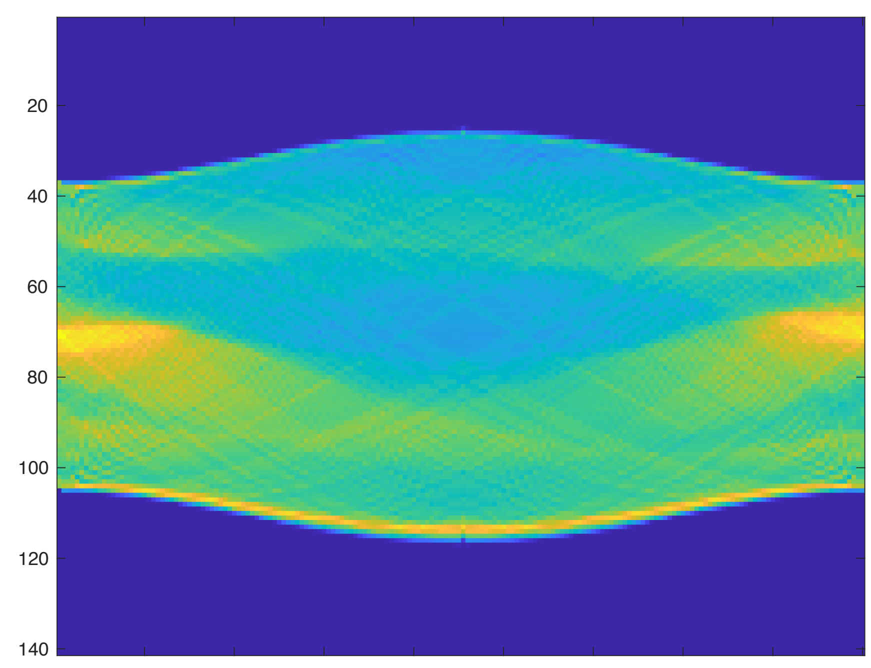
The Radon Transform



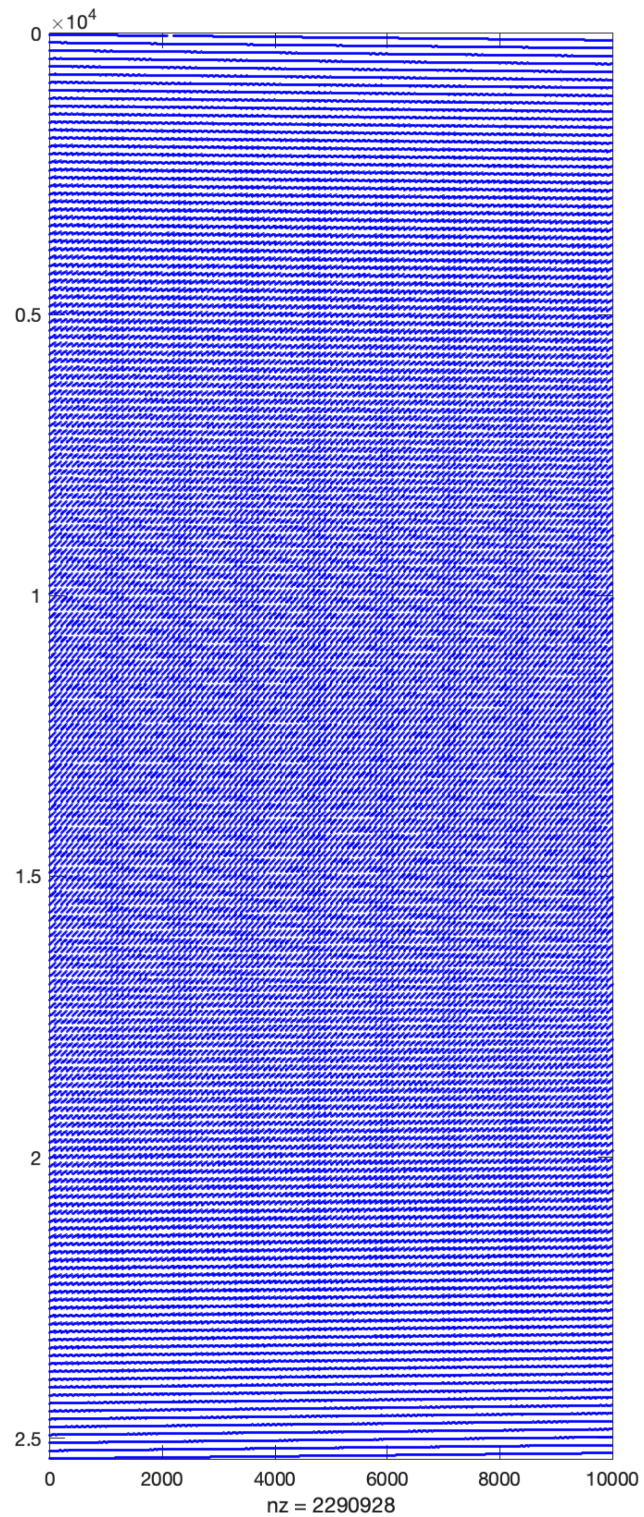
$$\left(R_{\theta} f^{(0)}\right)(s) = \int_{-\infty}^{+\infty} f^{(0)}(s\theta + t\theta^{\perp})dt.$$
$$= \int_{-\infty}^{\infty} f^{(0)}(s\cos\phi - t\sin\phi,\, s\sin\phi + t\cos\phi)dt.$$

$$\left(Rf^{(0)}\right)(s,\theta) = \left(R_{\theta} f^{(0)}\right)(s)$$





The “space-domain” operator



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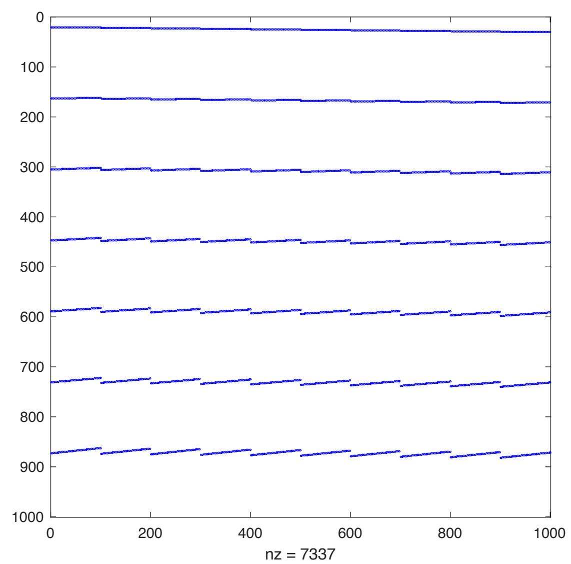
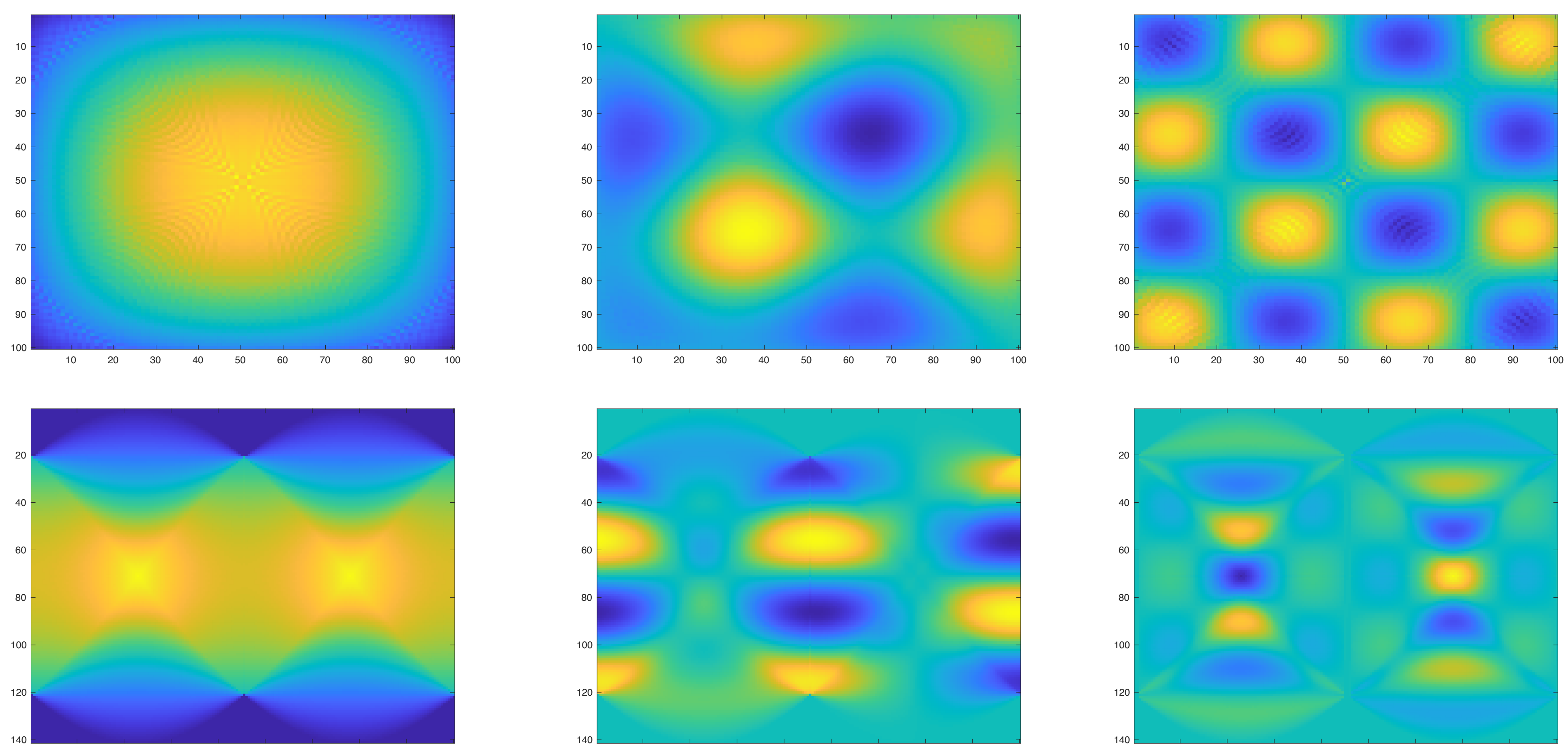
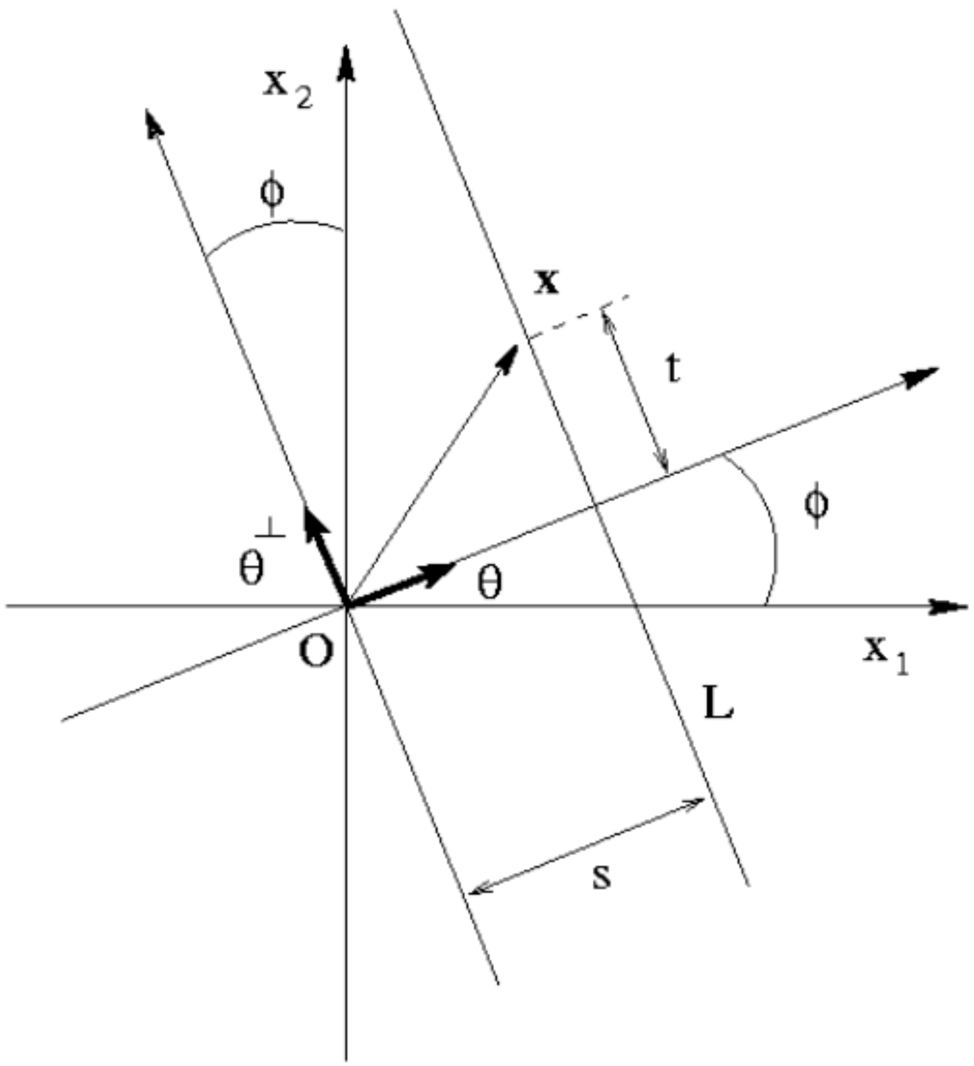


sing.values

Singular functions



Fourier representation



$$\boldsymbol{x} = s\boldsymbol{\theta} + t\boldsymbol{\theta}^{\perp}$$
$$s = \boldsymbol{\theta} \cdot \boldsymbol{x}$$

$$\left(R_{\boldsymbol{\theta}} f^{(0)}\right)(s) = \int_{-\infty}^{+\infty} f^{(0)}(s\boldsymbol{\theta} + t\boldsymbol{\theta}^{\perp})dt.$$

Fourier slice theorem: $(R_{\boldsymbol{\theta}} f)\hat{}(\omega) = \hat{f}(\omega\boldsymbol{\theta}).$

FT wrt s — FT wrt x,y

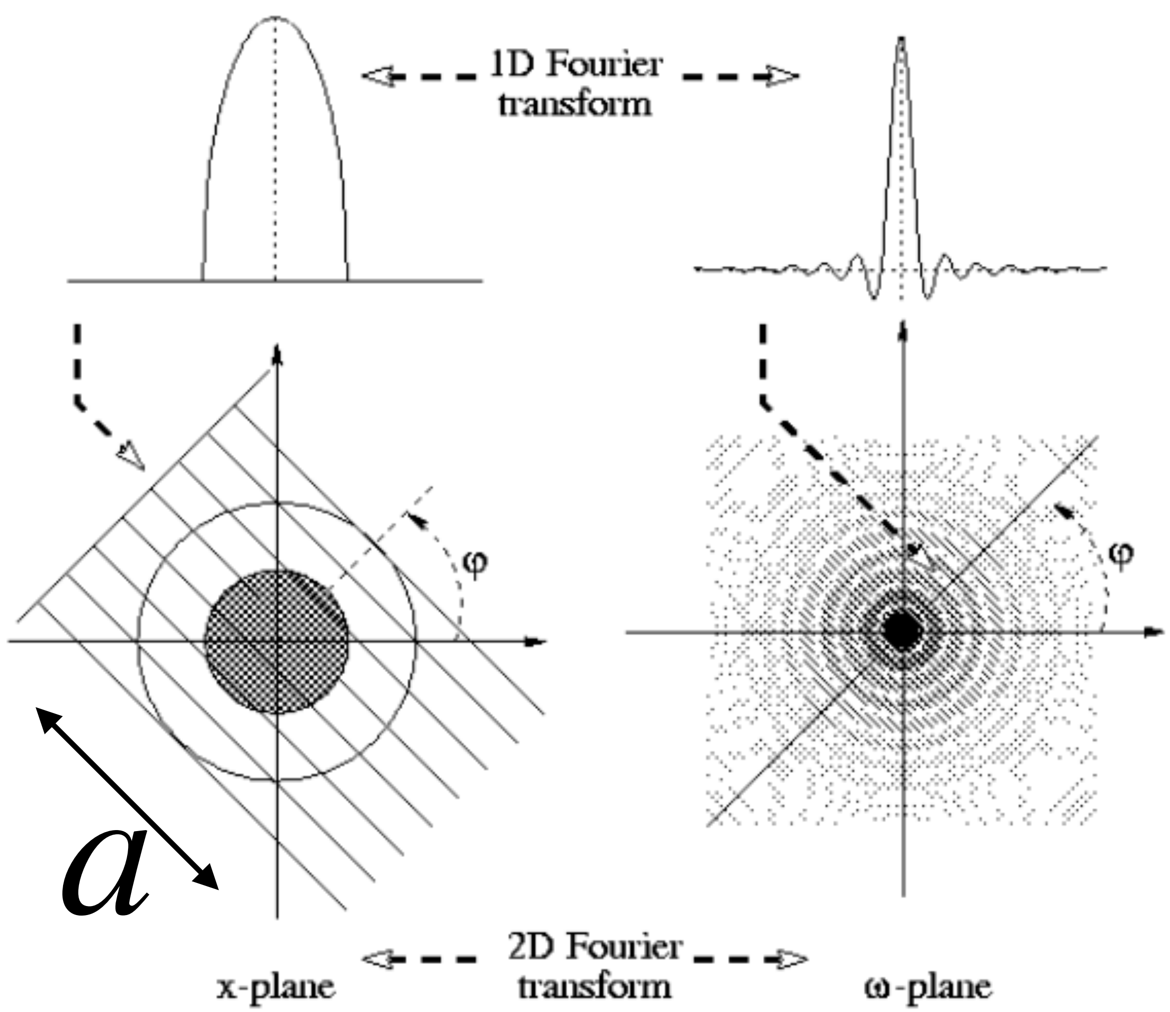
Proof:

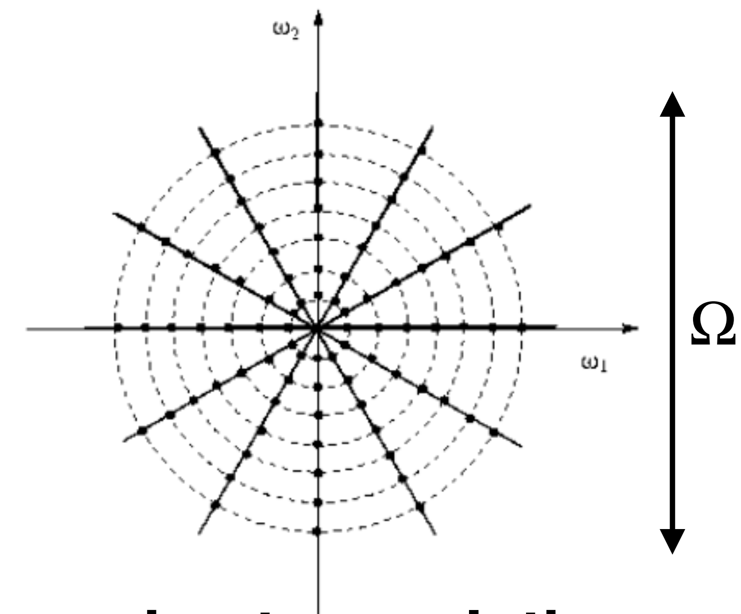
$$\begin{aligned}(R_{\boldsymbol{\theta}} f)\hat{}(\omega) &= \int_{-\infty}^{+\infty} e^{-i\omega s}\,(R_{\boldsymbol{\theta}} f)(s)ds \\ &= \int_{-\infty}^{+\infty} e^{-i\omega s}\left(\int_{-\infty}^{+\infty} f(s\boldsymbol{\theta} + t\boldsymbol{\theta}^{\perp})dt\right)ds. \\ &= \int f(\boldsymbol{x})\exp(-i\omega\boldsymbol{\theta}\cdot\boldsymbol{x})d\boldsymbol{x}\end{aligned}$$

Exercise: check with $\theta = (1{,}0)$

Fourier slice theorem

$$(R_\theta f)\hat{}(\omega) = \hat{f}(\omega\theta).$$





<u>**Approximate resolution analysis**</u>

- **Boundary points spacing (p=# projections)** $\Delta = \Omega\pi/p.$
- **Shannon (switch time&frequency): need** $\Delta = \pi/a$ **for correct sampling of FT**
- **Shannon (again): this gives resolution**

$$\delta \cong \frac{\pi}{\Omega} = \frac{\pi a}{p}.$$

- **Uniqueness**
- **Radial sampling of FT**
- **Reconstruction #1: resample + IFT**

There is much more…

- Stable reconstruction: Filtered Backprojection
- “Propagation of singularities” - micro-local analysis, Fourier Integral Operators,…
- Limited angle - problem becomes exponentially ill-posed

Band-limited systems

Bandlimited functions

$\operatorname{supp}\hat{f} \subset [-\Omega, \Omega]$, where $\Omega < \frac{\pi}{\tau}$.

$$f(x) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} e^{i\xi x}\left(\int_{\mathbb{R}} e^{-i\xi t} f(t)dt\right)d\xi$$
$$= \int_{\mathbb{R}} f(t)\frac{\sin\Omega(x-t)}{\pi(x-t)}dt.$$

Ideal low-pass filter:

$$P_\Omega f(x) = \int_{\mathbb{R}} \frac{\sin\Omega(x-t)}{\pi(x-t)} f(t)dt.$$

Convolution operators

$$g(y) = \int K\left(y - x\right) f(x)\, dx = (K \star f)(y), \quad y \in \mathbb{R}^d$$

- Spectral representation: $(Af)(\boldsymbol{y}) = \frac{1}{(2\pi)^d} \int \hat{K}(\omega)\hat{f}(\omega)e^{iy\cdot\omega}d\omega$
- Adjoint: $\left(A^* g\right)(x) = \int K^*\left(x' - x\right) g\left(x'\right) dx'$
- In general, not compact operator (on unbounded domains)
- Bandlimited assumption: $\operatorname{supp} \hat{K} \subset [-\Omega, \Omega]$ (e.g. $\chi_{[-\Omega,\Omega]}$)
- Inverse: $\left(A^{-1}g\right)(x) = \frac{1}{(2\pi)^d} \int \frac{\hat{g}(\omega)}{\hat{K}(\omega)} e^{ix\cdot\omega}d\omega$ - may not exist, or not continuous!
    - Compare with $A^{-1}y = \sum_{i=1}^{\infty} \frac{1}{\mu_i} \left\langle y, \psi_i \right\rangle \varphi_i$ if A is compact
- So, the inverse problem is still ill-posed

Example: image blur

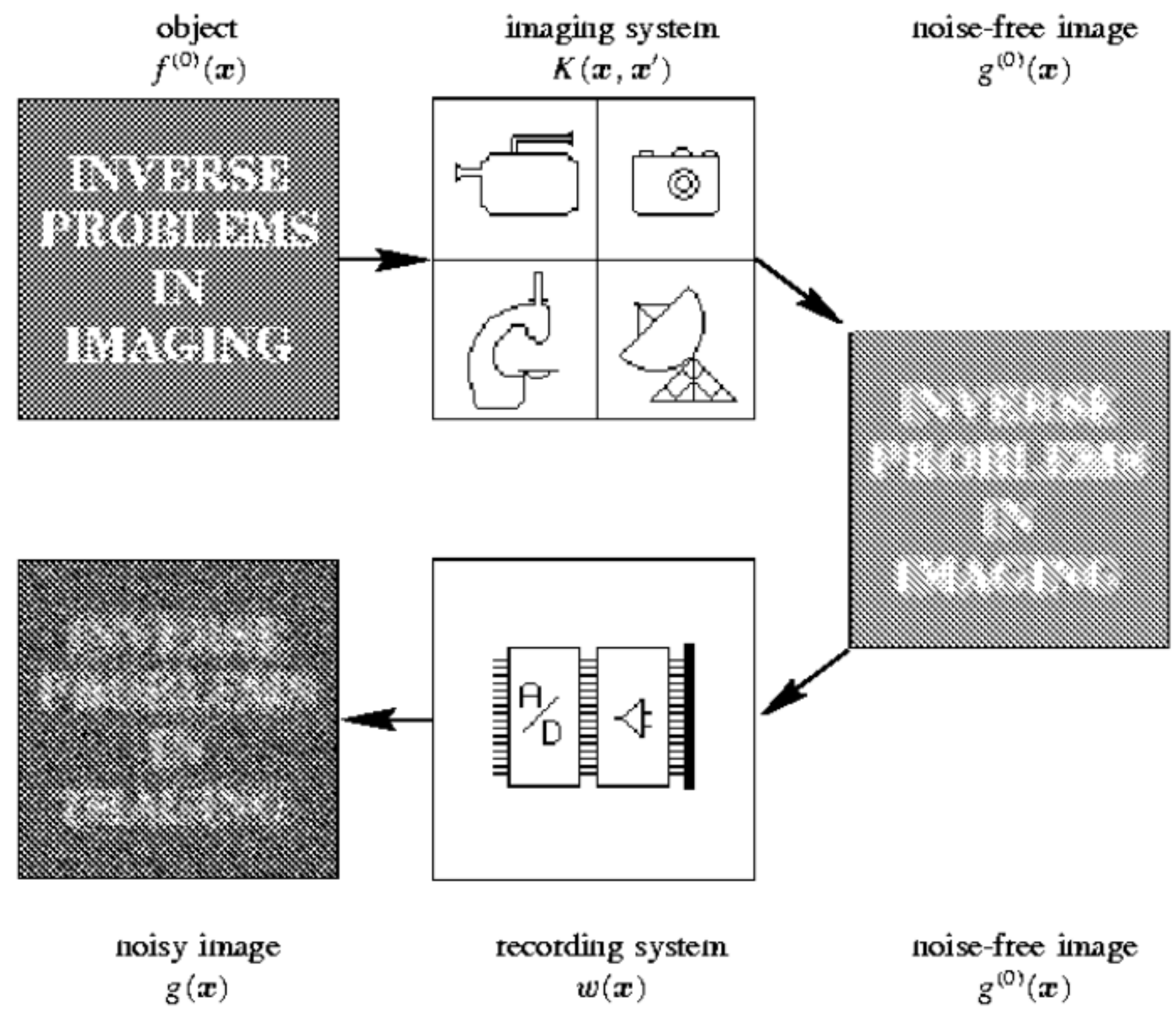
$$g(y) = \int K\left(y - x\right) f(x)\, dx = (K \star f)(y), \quad y \in \mathbb{R}^q$$

Point-spread function (PSF)

$$(Af)(\boldsymbol{y}) = \frac{1}{(2\pi)^q} \int \hat{K}(\omega)\hat{f}(\omega) e^{iy\cdot\omega} d\omega$$

Transfer function

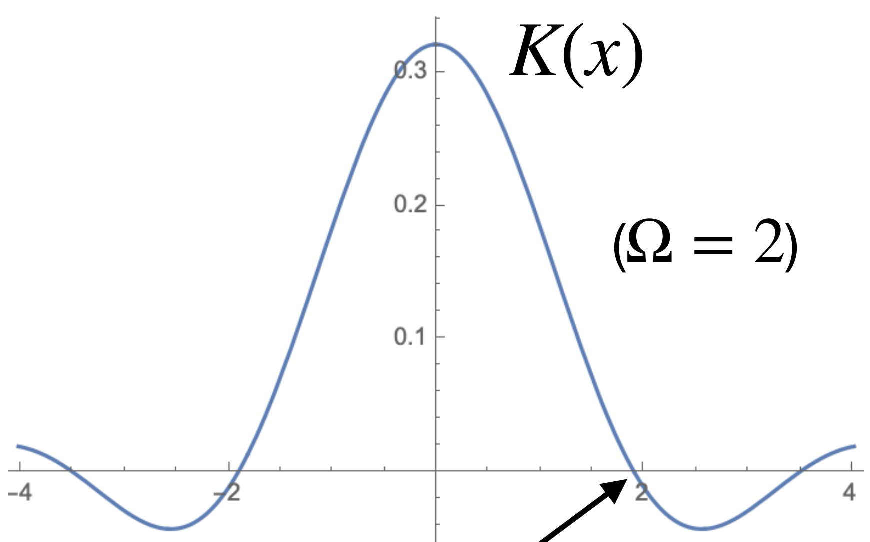
Motion blur: $K(\boldsymbol{x}) = \frac{1}{T}\int_0^T \delta[\boldsymbol{x} - \boldsymbol{a}(t)]dt$

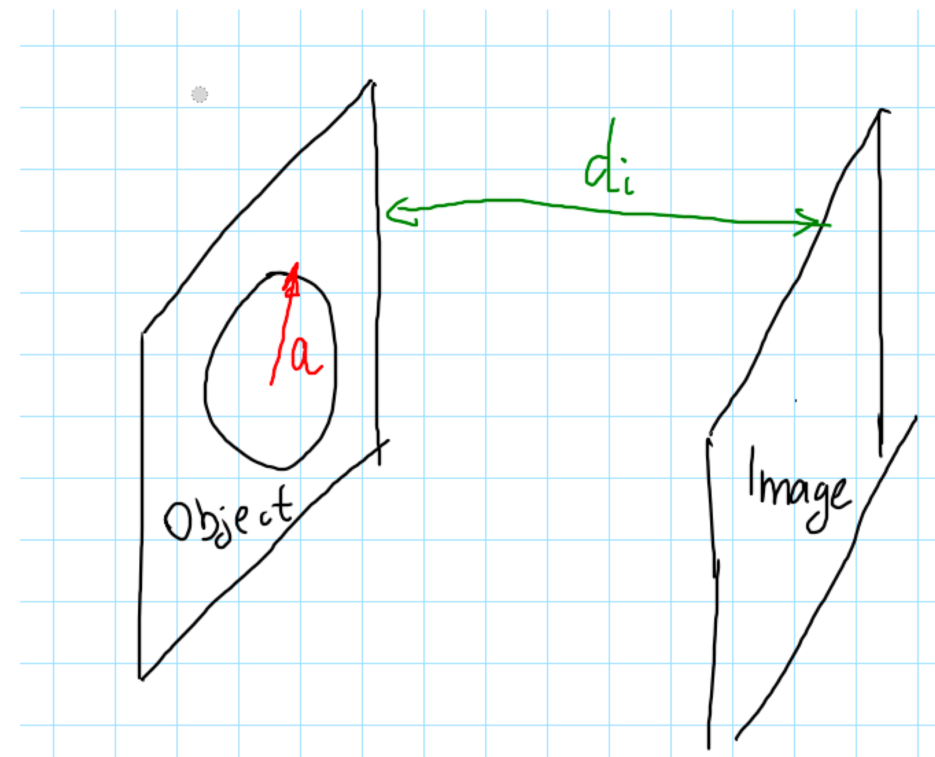


Example: diffraction

$$K(x) = \frac{\Omega}{2\pi}\frac{J_1(\Omega|x|)}{|x|} \qquad \Omega = \frac{2\pi}{\lambda}\frac{a}{d_i}$$

J_1 - Bessel function (1st kind)





(far field approximation)

first zero = $1.22\frac{\pi}{\Omega}$

Rayleigh resolution distance

J. W. Goodman, *Introduction to Fourier Optics*. Roberts and Company Publishers, 2005.

Direct regularisation

$$(Af)(y) = \frac{1}{(2\pi)^d}\int_{\mathscr{B}} \hat{K}(\omega)\hat{f}(\omega)e^{iy\cdot\omega}d\omega, \quad \operatorname{supp}\hat{K} = \mathscr{B}$$

- Directly in the frequency domain: $\hat{g}(\omega) = \hat{K}(\omega)\hat{f}(\omega)$
- Informal "eigenfunction" $\varphi_\omega = e^{i\omega t}, \quad \omega \in \mathscr{B}$
- Tikhonov: $\|Af - g\|_2^2 + \alpha^2\|f\|_2^2 = \|\hat{K}\hat{f} - \hat{g}\|_2^2 + \alpha_2^2\|\hat{f}\|_2^2$
  - Minimizer: $\hat{f}_\alpha(\omega) = \frac{\hat{K}^*(\omega)}{\alpha^2 + \left|\hat{K}(\omega)\right|^2}\hat{g}(\omega)$ (compare $x^{\delta,\alpha} = \sum_{i=1}^{\infty}\frac{\mu_i}{\mu_i^2+\alpha^2}\left\langle y^\delta, \psi_i\right\rangle \varphi_i$)
  - Other regularisers: e.g. $\|f''\|^2 \to \|\omega^2\hat{f}(\omega)\|_2^2$
- Landweber: $\hat{f}_{k+1}(\omega) = \frac{(1-(1-\tau|\hat{K}(\omega)|^2))^{k+1}}{\hat{K}(\omega)}\hat{g}(\omega)$
  - Compare $x^{k+1} = \sum_{i=1}^{\infty}\frac{(1-(1-\tau\mu_i^2))^{k+1}}{\mu_i}\langle y, \psi_i\rangle\varphi_i$

Bertero&Boccacci, Chapter 5

Convolution operators

$$\int K(t-s)f(s)ds = g(t)$$

- Periodic problems => circulant matrices
- Finite domains: Toeplitz matrices
- General case: Toeplitz operators

Periodic case

$$f(x) \sim \sum_{n\in\mathbb{Z}} c_n e^{inx}, \quad c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} f(x)dx$$

- **Fix** $n \in \mathbb{N}$ **. Define the Toeplitz matrix**

$$T_f^{(n)} = \begin{bmatrix} c_0 & c_1 & \cdots & c_n \\ c_{-1} & c_0 & \cdots & c_{n-1} \\ \ddots & & & \\ c_{-n} & c_{-n+1} & \cdots & c_0 \end{bmatrix}$$

- **For** $\boldsymbol{u} = (u_0, u_1, \ldots, u_n) \in \mathbb{C}^{n+1}$**, the corresponding quadratic form is:**

$$\langle \boldsymbol{u}, T_f^{(n)} \boldsymbol{u} \rangle = \sum_{i,j=0}^{n} c_{j-i} u_i \bar{u}_j$$

- **Put** $U(x) = \sum_{k=0}^{n} u_k e^{ikx}$**, then** $\langle \boldsymbol{u}, T_f^{(n)} \boldsymbol{u} \rangle = \int_0^{2\pi} \left| U(x) \right|^2 f(x)dx$
- **If** $T_f^{(n)}$ **was circulant then** $\mathrm{spec}(T_f^{(n)})$ **would be the DFT coefficients of** $(c_0, \ldots, c_n)$**, which should approximate the values of** $f\left(\frac{2\pi j}{n+2}\right)$**,** $j = 0, \ldots, n$
- **What happens when** $n \to \infty$**? (i.e.** $T_f^{(n)} \boldsymbol{u} \to \check{f} \star u \equiv (f \cdot U)^{\check{}}$**)**

Distribution and limiting density

- $\{\{a_1^{(n)}, \ldots, a_{n+1}^{(n)}\}\}_n$ and $\{\{b_1^{(n)}, \ldots, b_{n+1}^{(n)}\}\}_n$ are said to be **equidistributed** in $[-K, K]$ if $\forall F : [-K, K] \to \mathbb{R}$ continuous:

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n+1} F(a_j^{(n)}) - F(b_j^{(n)})}{n+1} = 0.$$

- Enough to check for **moments** $F_s(t) = t^s, \quad s = 0,1,\ldots,$

- Limiting "density" (measure) $\mu([\alpha, \beta]) = \lim_{n\to\infty} \frac{1}{n} \#\{j : a_j^{(n)} \in (\alpha, \beta)\}$

Szego's distribution theorem: I

$$f(x) \sim \sum_{n \in \mathbb{Z}} c_n e^{inx} \qquad T_f^{(n)} = \begin{bmatrix} c_0 & c_1 & \dots & c_n \\ c_{-1} & c_0 & \dots & c_{n-1} \\ \ddots & & & \\ c_{-n} & c_{-n+1} & \dots & c_0 \end{bmatrix}$$

- Assume $m < f(x) < M$ a.e. and $f \in L_1$ (Lebesgue)
- $\{\lambda_1^{(n)}, \dots, \lambda_{n+1}^{(n)}\}$ eigenvalues of $T_f^{(n)}$
- **Then:** $\{\{\lambda_1^{(n)}, \dots, \lambda_{n+1}^{(n)}\}\}_n$ and $\left\{ \{f\left(\frac{2\pi}{n+2}k\right)\}_{k=0}^{n+1} \right\}_n$ are equidistributed.
- So, $\operatorname{spec} T_f^{(n)}$ converges to the set of values of f.
- Can also estimate rates of convergence of extreme e.v.'s (resp. to m, M)
- Szego theory: consider system of polynomials on $\mathbb{D}$ (unit circle) orthogonal w.r.t. $f(x)$
- Generalisation to arbitrary curves in $\mathbb{C}$ (not just $\mathbb{D}$)

Grenander&Szegö, Chapter 5.2

Toeplitz operators

Toeplitz kernel: $K(s,t) = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} e^{i(s-t)x} f(x)dx = \left\{\mathscr{F}^{-1}f\right\}(s-t)$

Fact: the <u>spectrum</u> of $Ag = \int K(s-t)g(s)ds$ coincides with the <u>essential range</u> of f

Spectrum vs eigenvalues

Idea: approximate A by compact operators

$$A_T g = \int_0^T K(t-s)g(s)ds, \quad 0 < s < T.$$

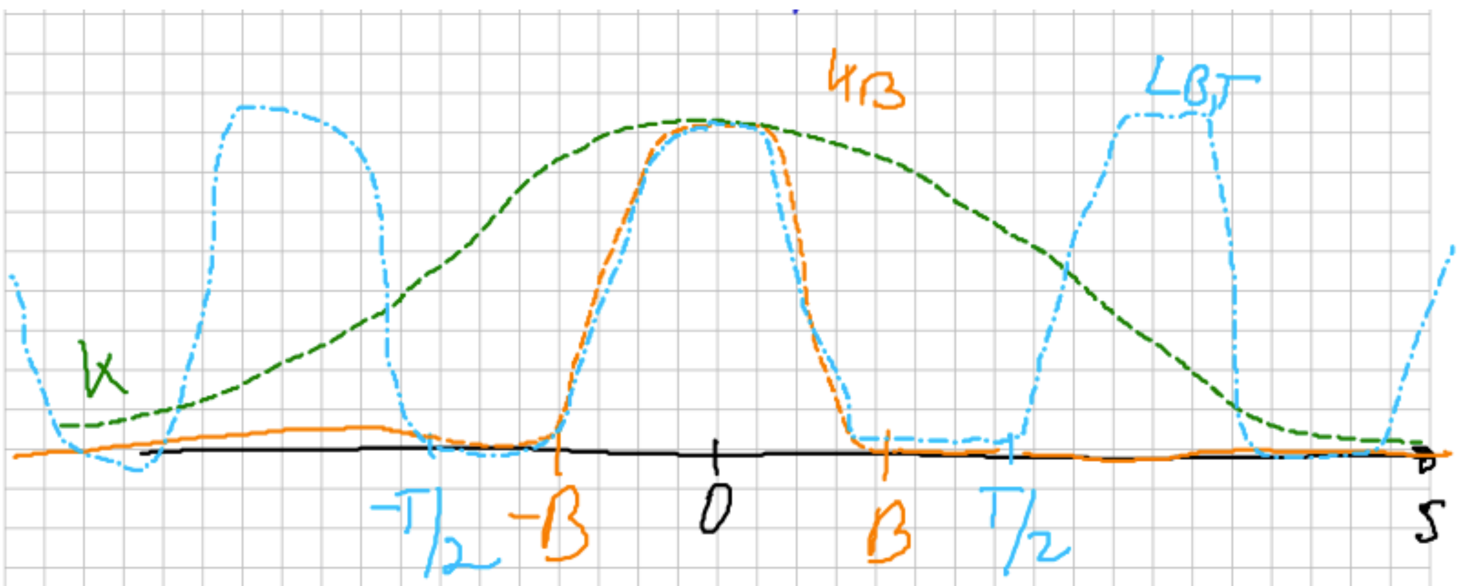
Compact approximation to Toeplitz operators

1. Introduce a parameter $0 < B < \frac{T}{2}$ and set

$$K_B(s) = \begin{cases} \left(1 - \frac{|s|}{B}\right) K(s) & |s| \leq B \\ 0 & \text{else}\,. \end{cases}$$

2. $f_B(x) = \left\{\mathcal{F}^{-1} K_B\right\}(x) = \left(\mathrm{sinc}^2 \frac{B\,\cdot}{\pi} \star f\right)$

$L_{B,T}(s) = K_B(s)$ **for** $|s| \leq \frac{T}{2}$ **(periodic)**



Easy to check: $\psi_n(t) = e^{2\pi i n \frac{n}{T}}$ **is an eigenfunction of** $A_{L_{B,T}}$ **with eigenvalue** $\lambda_n = f_B\left(\frac{2\pi}{T} n\right), \quad n = 0, \pm 1, \ldots$

Take $B, T \to \infty$

Szego's distribution theorem: II

- $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, real-valued (filter)
- $K(t) = \left\{\mathscr{F}^{-1} f\right\}(t)$
- $A_T g = \int_0^T K(t-s) g(s) ds, \quad 0 < s < T$

Then: the eigenvalues of A_T have asymptotic distribution of $(2\pi)^{-1} f(x)$, i.e.

$$\lim_{T \to \infty} \frac{1}{T} \# \left\{\lambda \in \text{spec}(A_T) : \ \lambda \in (a,b)\right\} = (2\pi)^{-1} m \left\{x : \ f(x) \in (a,b)\right\}$$

↑ *(Lebesgue measure)*

Grenander&Szegö, Chapter 8.6

Super-resolution in band-limited systems

The super-resolution question: linear setting

- Unknown: $f \in L^2[-T, T]$
- Observe: Ω-bandlimited version of f:
$$g = P_\Omega f = \int_{-T}^{T} \frac{\sin(\Omega(t-s))}{\pi(t-s)} f(s)ds$$
- Define $A_{\Omega,T} : L^2[-T, T] \to L^2(\mathbb{R})$ by $A_{\Omega,T} f = P_\Omega f$
- Need to solve ill-posed inverse problem with $A_{\Omega,T}$
- This problem is so important in applications as to warrant detailed analysis

Slepian operator

- $A_{\Omega,T}f = \int_{-T}^{T} \frac{\sin(\Omega(t-s))}{\pi(t-s)} f(s)ds,\ s \in \mathbb{R}$
- Adjoint: $(A^*_{\Omega,T}f)(s) = \int_{\mathbb{R}} f(t)\mathrm{sinc}(\Omega(s-t)/\pi)dt,\ |s| < T$
- Set $B_{\Omega,T} = A^*_{\Omega,T}A_{\Omega,T}$
- Exercise: $B_{\Omega,T}f(t) = \int_{-T}^{T} \frac{\sin(\Omega(t-s))}{\pi(t-s)} f(s)ds,\ |t| < T$
- Space-bandwidth product: $c = \Omega T$
- $K_c(t) = \frac{c}{\pi}\mathrm{sinc}(ct/\pi) = \frac{\sin(ct)}{\pi t}$

$$(B_c f)(t) = \int_{-1}^{1} \frac{\sin(c(t-s))}{\pi(t-s)} f(s)ds,\ |t| < 1.$$

Questions

$$(B_c f)(t) = \int_{-1}^{1} \frac{\sin(c(t-s))}{\pi(t-s)} f(s)ds, \; |t| < 1.$$

- Decay of eigenvalues
- Eigenvalue distribution
- Eigenfunctions

What do we expect?

$$(B_c f)(t) = \int_{-1}^{1} \frac{\sin(c(t-s))}{\pi(t-s)} f(s)ds, \; |t| < 1.$$

By Szego's theorem II, we should get a step in the limit $c \to \infty$

The sinc **kernel is analytic, so should expect super-exponential decay in** n

Exponential convergence to extreme eigenvalues (0 and 1)

$$(B_c f)(t) = \int_{-1}^{1} \frac{\sin(c(t-s))}{\pi(t-s)} f(s)ds, \ |t| < 1.$$

- Widom 1964: $\lambda_n \approx \dfrac{c^{2n+1}}{(n!)^2}, \quad n \to \infty$ (c fixed)
- Fuchs 1964: $1 - \lambda_n \sim 4\sqrt{\pi c}\dfrac{8^n c^n}{n!}\mathrm{e}^{-2c}, \ c \to \infty$ (n fixed)

What about the middle?

$$(B_c f)(t) = \int_{-1}^{1} \frac{\sin(c(t-s))}{\pi(t-s)} f(s)ds, \ |t| < 1.$$

- Fix $\alpha \in \left(0, \frac{1}{2}\right)$, how many e.v.'s are between α and $1-\alpha$?
- By Mercer's theorem, $m_1 = \sum_k \lambda_k = \text{trace}(B_c) = \int K_c(x,x)dx$. Direct computation: $m_1 = \dfrac{2c}{\pi}$
- Also, $m_2 = \sum_k \lambda_k^2 = \|B_c\|^2 = \iint_{[-1,1]^2} K_c^2(x-y)dxdy$, can estimate this directly as well: $m_2 \geq \frac{2c}{\pi} - c_1 \log c - c_2$ for some c_1, c_2
- So $\sum_{\alpha<\lambda<1-\alpha} \lambda(1-\lambda) \geq \alpha(1-\alpha)\#\{\lambda : \alpha < \lambda < 1-\alpha\}$
- $\sum_{\alpha<\lambda<1-\alpha} \lambda(1-\lambda) \leq \sum_\lambda \lambda(1-\lambda) = m_1 - m_2 \leq c_1 \log c + c_2$

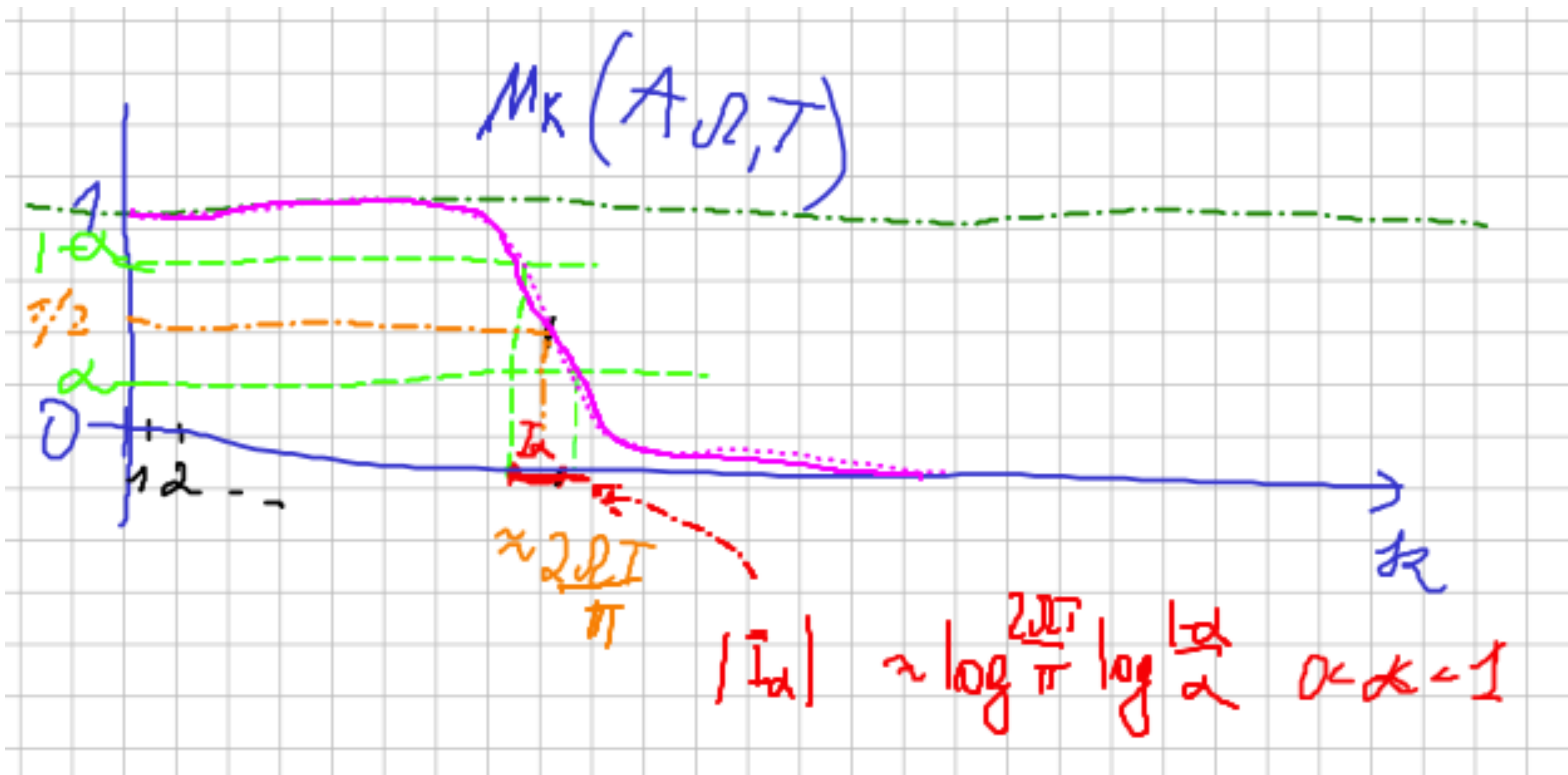
$$\#\{\lambda : \alpha < \lambda < 1-\alpha\} \leq \frac{c_1 \log c + c_2}{\alpha(1-\alpha)}$$

"Plunge region"

Distribution result

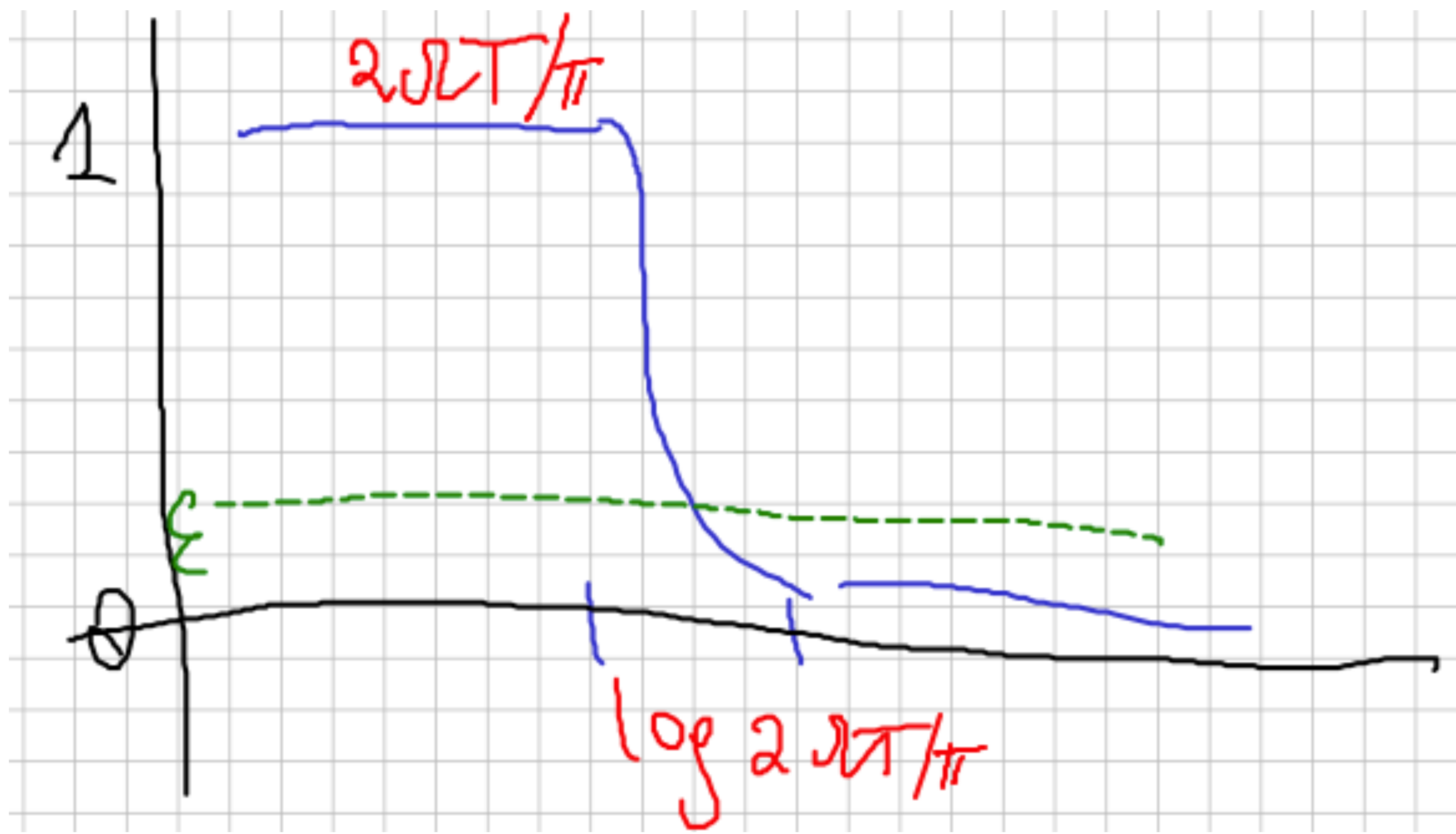
$$\#\{\lambda : \lambda > \alpha\} = \frac{2c}{\pi} + \frac{1}{\pi^2} \log \frac{1-\alpha}{\alpha} \log \frac{2c}{\pi} + o(\log c) \quad 0 < \alpha < 1,\ c \to \infty$$

Landau&Widom 1980



- They prove the result for general case of $[-\Omega, \Omega]$ and $[-T, T]$ replaced with unions of intervals C, S (then $c \propto m(C)m(S)$)

Implications for SR



- ΩT is precisely the number of points in $[-T, T]$ taken at Nyquist rate
- As $T \to \infty$, the "super-resolution factor" vanishes ($\approx O(\log T/T)$)
- For fixed $T \gg 1$, we expect $\log \frac{1-\epsilon}{\epsilon} \approx \log \frac{1}{\epsilon}$ (as $\epsilon \to 0$) extra degrees of freedom
- In practice, may obtain significant SR for small c (i.e. T)

"$2\Omega T$" theorem

- Signal $r(t)$ is "time-limited to $[-T/2, T/2]$ at level ϵ" if $\int_{|t|>T/2} r^2(t)dt < \epsilon$
- Signal is "band-limited to $[-\Omega, \Omega]$ at level ϵ" if $\int_{|\omega|>\Omega} \hat{r}^2(\omega)d\omega < \epsilon$
- Set of signals S "has approximate dimension $N(T, \epsilon)$ on $[-T/2, T/2]$ at level ϵ" if $\exists\{\varphi_1, \ldots, \varphi_N\}$ which can approximate any $f \in S$ with error $\leq \epsilon$, and N is the smallest such number

Theorem (Slepian 1976): let S_ϵ be the set of signals time-limited and band-limited at level ϵ. If $N = N(\Omega, T, \epsilon', \epsilon)$ is the approximate dimension of S_ϵ at level ϵ', then $\forall \epsilon' > \epsilon$: $\lim_{T\to\infty} N/T \to 2\Omega$, and $\lim_{\Omega\to\infty} N/\Omega \to 2T$.

Singular functions of $A_{\Omega,T}$

- Turns out these are well-known special functions: **prolate spheroidal wave functions (PSWFs)**
- They are the solutions of the wave equation in a certain system of coordinates
- They have very useful properties
- By now, part of the standard toolbox in computational and applied mathematics

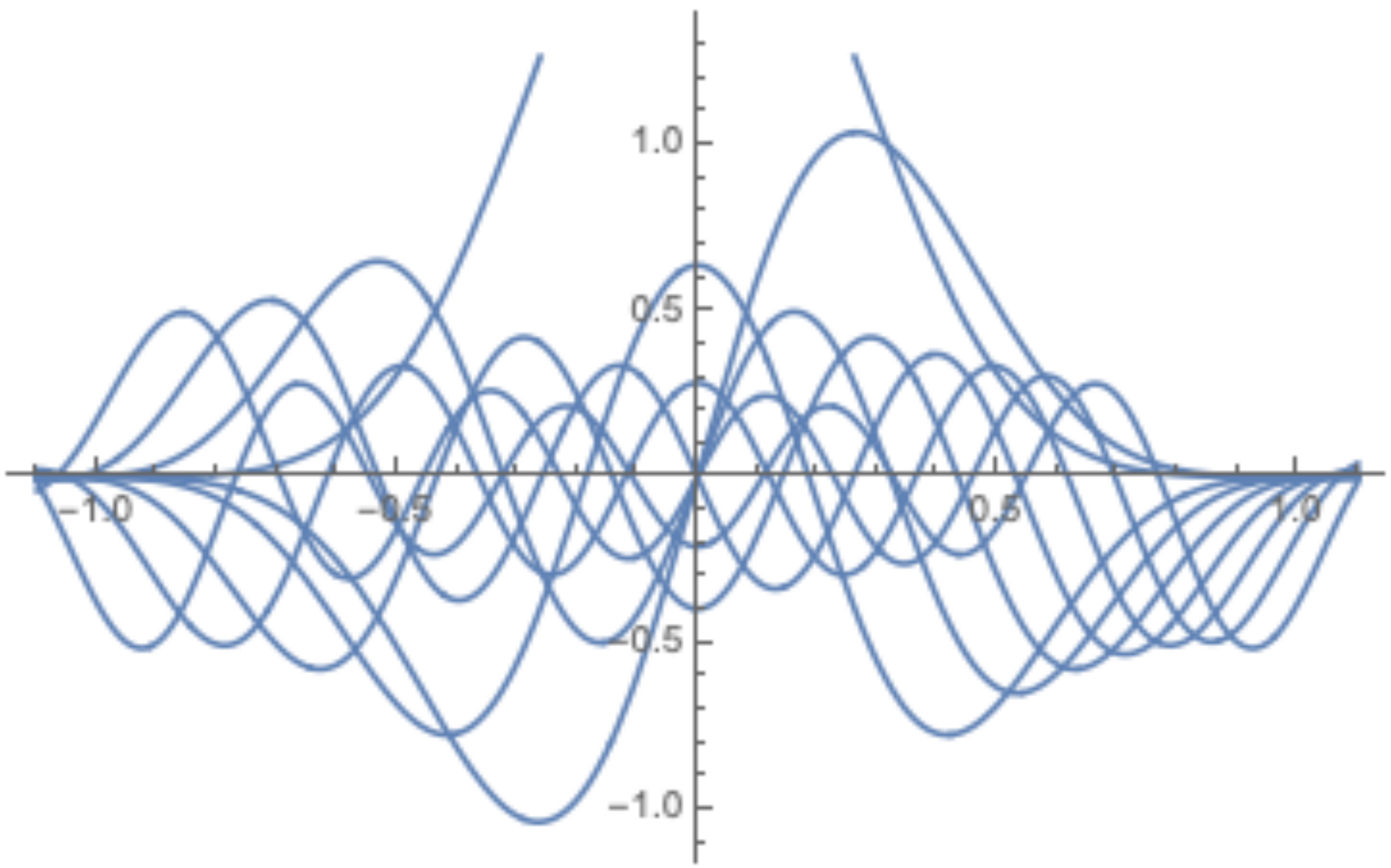
PSWFs

Sturm-Liouville theory

- $\frac{d}{dx}(1-x^2)\frac{d\psi(c;x)}{dx} + (\chi - c^2x^2)\psi(c;x) = 0$
- $\int_{-1}^{1} \frac{\sin(c(t-s))}{\pi(t-s)}\psi_n(c;s)ds = \lambda_n\psi_n(c;t)$
- Can be extended to $t \in \mathbb{R}$ by the above formula
- Simultaneously orthogonal on $\mathbb{R}$ and on $[-1,1]$
- $\widehat{\psi_n}$ is a rescaled copy of ψ_n
- Complete in $L^2[-1,1]$ and in PW_c (space of c-bandlimited functions)
- ψ_n has exactly n zeros in $[-1,1]$
- Can be computed efficiently
- ψ_0 solves the "spectral concentration problem" (the Ω-bandlimited signal mostly concentrated on $[-T/2,T/2]$)
- Useful for extrapolation

```
In[28]:= Plot[Table[SpheroidalPS[k, 0, 15, t], {k, 0, 8}], {t, -1.1, 1.1}]
```

Out[28]=



Gerchberg-Papoulis method

- Recall:

$$A_{\Omega,T}f = \int_{-T}^{T} \frac{\sin(\Omega(t-s))}{\pi(t-s)} f(s)ds : L^2([-T,T]) \to L^2(\mathbb{R})$$

- $P_\Omega, P^{(T)} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ - frequency & time-limiting projection operators

- Define $\tilde{A}_{\Omega,T} := P_\Omega P^{(T)} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$

- Super-resolution problem: solve $\tilde{A}_{\Omega,T} f = g$

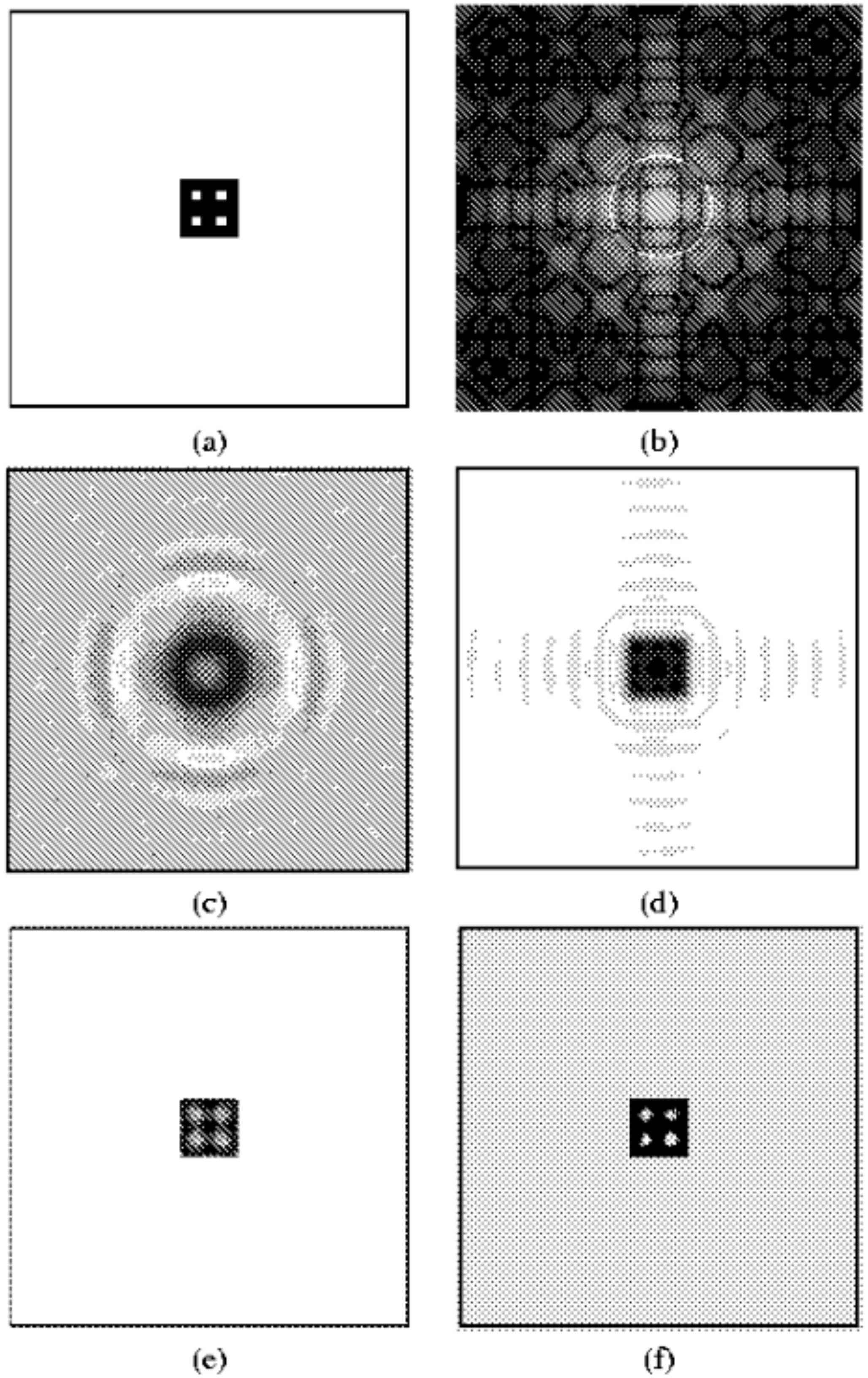
Gerchberg-Papoulis method - cntd.

- $\tilde{A}_{\Omega,T} := P_\Omega P^{(T)} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$
- Adjoint: $\tilde{A}^*_{\Omega,T} = P^{(T)} P_\Omega$
- $\tilde{A}^*_{\Omega,T} \tilde{A}_{\Omega,T} = P^{(T)} P_\Omega P^{(T)} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$
- Landweber: $f_{k+1} = f_k + \tau \tilde{A}^*_{\Omega,T}(g - \tilde{A}_{\Omega,T} f_k)$
- Can choose $0 < \tau < 2$, e.g. $\tau = 1$
- Suppose $f_0 = 0$ and $g = P_\Omega g$ (initial data is band limited)
- Can show by induction that $\forall k : P^{(T)} f_k = f_k$
- Further manipulations yield: $f_{k+1} = P^{(T)} \left(g + (I - P_\Omega) f_k \right)$

G-P: compute via Fourier transforms

1. $f_1 = P^{(T)} g$
2. **Repeat:**
   - **A.** $f_k \to \widehat{f}_k$
   - **B.** $\widehat{g}_{k+1} = \widehat{g} + (1 - \chi_\Omega) \widehat{f}_k$
   - **C.** $\widehat{g}_{k+1} \to g_{k+1}, f_{k+1} = P^{(T)} g_{k+1}$

Example



1. $$f_1 = P^{(T)} g$$
2. **Repeat:**
   - **A.** $$f_k \rightarrow \hat{f}_k$$
   - **B.** $$\hat{g}_{k+1} = \hat{g} + (1 - \chi_\Omega)\hat{f}_k$$
   - **C.** $$\hat{g}_{k+1} \rightarrow g_{k+1}, f_{k+1} = P^{(T)} g_{k+1}$$

Positivity constraint:

replace $P^{(T)}$ with $\tilde{P} = P_+ P^{(T)} = P^{(T)} P_+$

Figure 11.6. Example of super-resolution in far-field acoustic holography. (a) The object: a grid $2.5\lambda \times 2.5\lambda$ with bars 0.5λ wide. (b) The modulus of the FT of the object in (a); the white circle indicates the band of the far-field data. (c) The modulus of the noisy image at the distance 5λ from the object plane. (d) The reconstruction obtained by means of the inverse filtering method. (e) The reconstruction obtained by means of the iterative method with the constraint of bounded support. (f) The reconstruction obtained by means of the iterative method with the constraints of bounded support and positivity.

Bertero&Boccacci, Chapter 11

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