Inverse Problems - Lecture 5

0372402501 - Topics in Inverse Problems and Super Resolution, Fall 2022/23

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Fourier analysis - refresher

Fourier series

$f(x) : [0, 2\pi) \to \mathbb{R}$ (or $\mathbb{C}$)

- Analysis: $c_n := \frac{1}{2\pi} \int_0^{2\pi} e^{-\imath nx} f(x) dx$ (n-th Fourier coefficient)
- Synthesis: $f(x) \sim \sum_{n \in \mathbb{Z}} c_n e^{\imath nx}$
- Smoothness vs. decay:
  - jumps: $c_n \sim 1/n$
  - $f \in C^0 \to c_n \sim 1/n^2$ (e.g. corners)
  - $f \in C^d \to c_n \sim n^{-d-2}$
  - f analytic $\to c_n \sim r^n$, for $0 < r < 1$
- Riemann-Lebesgue lemma: $f \in L^1 \implies c_k \to 0$.

Convergence

$$f(x) \sim \sum_{n\in\mathbb{Z}} c_n e^{\imath nx}, \quad c_n := \frac{1}{2\pi}\int_0^{2\pi} e^{-\imath nx} f(x)dx$$

- Global
  - $f \in L^1 \rightarrow$ can diverge everywhere
  - $f \in L^2 \rightarrow$ converges a.e. in mean square sense
  - $f \in C^\alpha \rightarrow$ uniform convergence everywhere
  - $f \in BV \rightarrow$ convergence everywhere
- Local
  - f differentiable at $x \rightarrow$ converges at x
  - f has a finite jump $\rightarrow$ convergence to midpoint
  - Dini conditions...

Dirichlet kernel

$$f(x) \sim \sum_{n\in\mathbb{Z}} c_n e^{\imath nx}, \quad c_n := \frac{1}{2\pi}\int_0^{2\pi} e^{-\imath nx} f(x)dx$$

- $\delta_N(x) = \frac{1}{2\pi}\sum_{n=-N}^{N} e^{\imath nx} = \frac{1}{2\pi}\frac{\sin(N+1/2)x}{\sin x/2}$

$$f_N(x) = \sum_{k=-N}^{N} c_k e^{\imath kx} = \frac{1}{2\pi}\sum_{k=-N}^{N} e^{\imath kx}\int_0^{2\pi} f(t)e^{-\imath kt}dt$$
$$= \int_0^{2\pi}\delta_N(y)f(x-y)dy = (\delta_N * f)(x)$$

- Weak convergence: $\int \delta_N(x)f(x)dx = c_{-N} + \cdots + c_N \longrightarrow f(0)$.

Dirichlet kernel

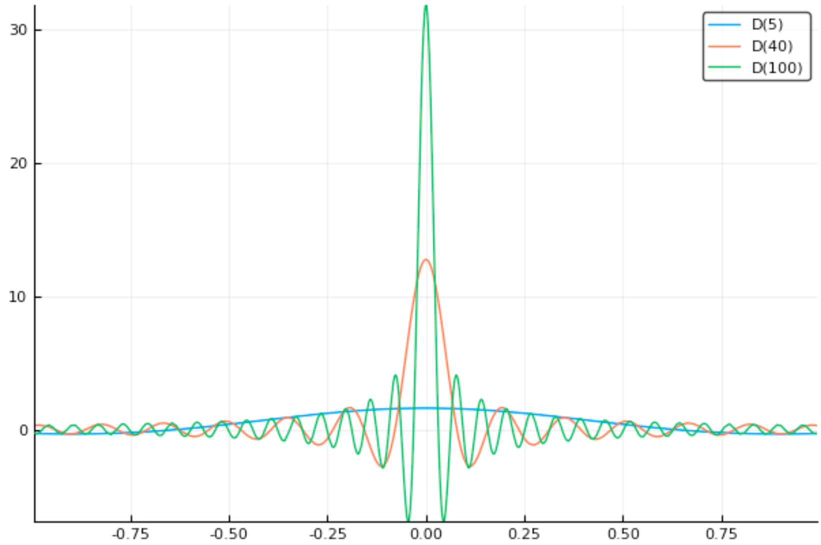


Figure 1: The Dirichlet kernel

Gibbs oscillations

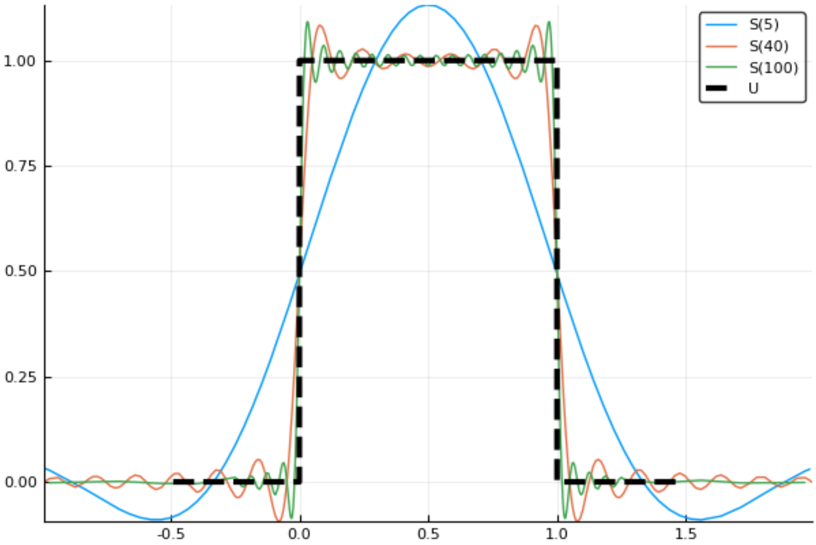
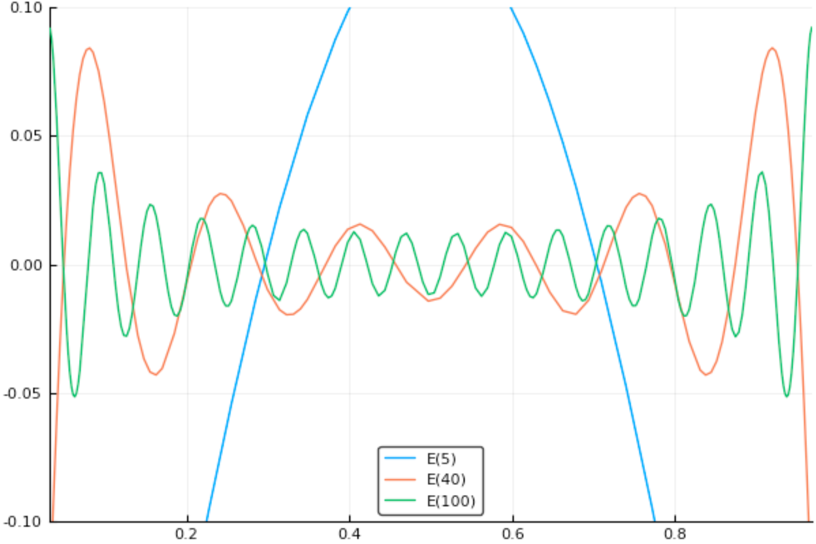


Figure 2: Gibbs phenomenon

Approximation error



$$\|E_n\|_2 \sim n^{-\frac{1}{2}}$$

Analysis

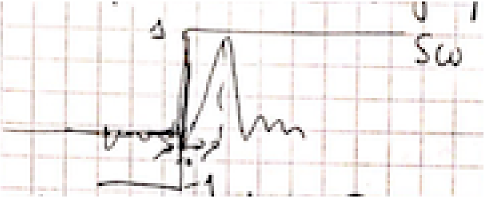


Figure 3: Analysis

- Choose x_0 for which $\delta_N(x_0) = 0 \rightarrow x_0 = \frac{\pi}{N+1/2}$
- As $N \rightarrow \infty$, sidelobes contribute ≈ 0 (cancel)
- From the main lobe:

$$J = \int_{-\pi}^{\pi} SW(t)\delta_N(x_0 - t)dt \rightarrow \int_{-x_0}^{x_0} \delta_N(t)dt$$

$$J \approx \frac{\pi}{N+1/2}\frac{1}{2\pi}\int_{-1}^{1} \frac{\sin \pi y}{\sin \frac{\pi y}{2(N+1/2)}}dy \approx \int_{-1}^{1} \frac{\sin \pi y}{\pi y}dy$$

$$\approx 1.17898$$

Convolution

$f(x) \sim \sum \hat{f}(k)e^{\imath kx}$, $g(x) \sim \sum \hat{g}(k)e^{\imath kx}$ periodic

$$h(x) = \int f(t)g(x-t)dt = \cdots = 2\pi \sum \hat{f}(k)\hat{g}(k)e^{\imath kx}$$

$$\hat{h}(k) = 2\pi \hat{f}(k)\hat{g}(k)$$

- $(c \star d)_n = \sum_{k\in\mathbb{Z}} c_k d_{n-k} \rightarrow f(x)g(x) = \sum_{n\in\mathbb{Z}} (c \star d)_n e^{\imath nx}$
- $\left(\sum c_k e^{\imath kx}\right)' = \sum (\imath k) c_k e^{\imath kx}$
- $\int \left(\sum c_k e^{\imath kx}\right) dx = \sum_{k\neq 0} \frac{c_k}{\imath k} e^{\imath kx}$

Discrete signals

- $\{x[n]\}_{n=\infty}^{\infty}$, e.g. samples of a continuous-time signal
- Discrete-Time Fourier Transform ("dual Fourier series")

$$X_{2\pi}(\omega) = \sum_{n=\infty}^{\infty} x[n] e^{-in\omega}$$

- Exists if $x[n]$ absolutely summable
- Inversion formula:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_{2\pi}(\omega) e^{in\omega} d\omega$$

Signals and systems

$$x = \sum x[k]\delta[n-k], \quad \delta = \text{unit impulse}$$

- Linear system (filter):

$$\begin{cases} y[n] = (Tx)[n] = \sum y[k]\delta[n-k] = \sum_k x[k]T\{\delta[n-k]\} \\ T\{\delta[n-k]\} \equiv h_k[n] \end{cases}$$

- Time-invariant system: $h_k[h] = h_0[h-k]$ (impulse response)
- Switching between time and frequency in the convolution thm:

$$\sum_{n=\infty}^{\infty} (x \star h)_n e^{-\imath nx} = X_{2\pi}(\omega) \qquad \underbrace{H_{2\pi}(\omega)}_{\text{frequency response}}$$

- Convolution (time/freq) $\leftarrow \rightarrow$ multiplication (freq.time)

Discrete Fourier Transform

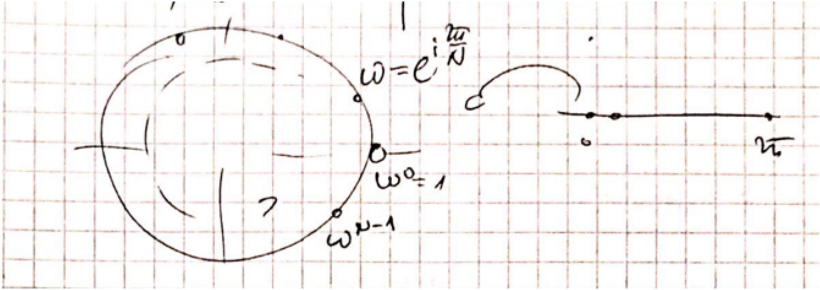


Figure 4: DFT

- divide $[0, 2\pi]$ into N equal intervals, $f \equiv \left\{f\left(k\frac{2\pi}{N}\right)\right\}_{k=0}^{N-1} \in \mathbb{C}^N$
- DFT basis: $\left\{e^{ik\bar{x}}\right\}_{k=0,\dots,N-1}$ where $\bar{x} = \{x_j\}_{j=0}^{N-1} = \left\{j\frac{2\pi}{N}\right\}_{j=0}^{N-1}$
- Approximate $c_k(f) = \frac{1}{2\pi}\int_0^{2\pi} e^{-\imath nx} f(x)dx$ by

$$\widehat{f}_N = \frac{1}{2\pi}\sum_{j=0}^{N-1} f\left(j\frac{2\pi}{N}\right) e^{-\imath kj\frac{2\pi}{N}} \cdot \frac{2\pi}{N} = \frac{1}{N}\sum_{j=0}^{N-1} f\left(\frac{j}{N}2\pi\right)\omega^{-jk}$$

DFT and FFT

- $\hat{f}_N = \frac{1}{N} F_N^* f_N$ where

$$F_N = \begin{bmatrix} \omega^0 & \omega^0 & \dots & \dots & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \dots & \omega^{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \omega^0 & \omega^{N-1} & \omega^{2(N-1)} & \dots & \omega^{(N-1)^2} \end{bmatrix}, \; \omega = e^{\imath \frac{2\pi}{N}}, \omega^N = 1.$$

- $F_N^* F_N = N \cdot I_N$ (in matlab, **fft** = multiplication by F_N^*)
- Fast algorithm (FFT), complexity: $\sim \frac{1}{2} N \log_2 N$

Convolutions & DFT

$$P = \begin{bmatrix} 0 & 0 & \dots & \dots & 1 \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \qquad \text{(cyclic shift)}$$

- For $x \in \mathbb{C}^N$, define the circulant matrix

$$\mathbf{X} = \left[x \; Px \; \dots \; P^{N-1}x \right] = \begin{bmatrix} x_0 & x_{N-1} & \dots & x_1 \\ x_1 & x_0 & \dots & x_2 \\ \ddots & \ddots & \ddots & \ddots \\ x_{N-1} & x_{N-2} & \dots & x_0 \end{bmatrix}$$

- Let $\mathbf{c} = \frac{1}{N} F_N^* x \to \left(\frac{1}{N} F_N^* P x \right)_k = \omega^{-k} c_k$
- $\mathbf{X} = \underbrace{F_N}_{\text{ifft}} \underbrace{\operatorname{diag}(c_0, \dots, c_{N-1})}_{\text{fft}(x)} \underbrace{F_N^*}_{\text{fft}}$

Filters

$$\mathbf{X} = \underbrace{F_N}_{\text{ifft}} \underbrace{\mathrm{diag}(c_0, \ldots, c_{N-1})}_{\text{fft}(x)} \underbrace{F_N^*}_{\text{fft}}$$

- $d \in \mathbb{C}^N \to \mathbf{X}d$ is a circular/cyclic convolution

$$(x \otimes d)_n = \sum_{k=0}^{N-1} x_{n-k} d_k = \sum_{j=0}^{N-1} x_j d_{n-j}$$

- If both x, y are finite signals, we can use the above to compute $(x \star y)$ by zero-padding

Fourier transform

- Start with Fourier series on $[-T/2, T/2]$:

$$f(x) \sim \frac{1}{T}\sum_k c_k e^{\imath k \frac{2\pi}{T} x}, \quad c_k = \int_{-T/2}^{T/2} f(t) e^{-\imath \frac{2\pi}{T} kt} dt$$

- Define $\xi = \frac{2\pi k}{T}$, $d\xi = \frac{2\pi}{T}$ and take $T \to \infty$: (need $f \in L^1$)

$$\begin{cases} \hat{f}(\xi) = \int_{-\infty}^{\infty} f(t) e^{-i\xi t} dt \\ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi \end{cases}$$

- Convolution:

$$f \star g = \int_{\mathbb{R}} f(x-y) g(y) dy, \quad \widehat{(f \star g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$$

- $\widehat{\partial_x f}(\xi) = \imath\xi\hat{f}(\xi)$, $\widehat{(\int^x f)}(\xi) = \frac{1}{\imath\xi}\hat{f}(\xi)$, $f\widehat{(x-d)}(\xi) = e^{-\imath\xi d}\hat{f}(\xi)$

Sinc function

$$\hat{f}(\omega) = \chi_{[-\Omega,\Omega]} = \begin{cases} 1 & |\omega| \leqslant \Omega \\ 0 & \text{else} \end{cases}$$

$$\rightarrow f(t) = \tfrac{1}{2\pi} \int_{-\Omega}^{\Omega} e^{\imath \xi t} d\xi = \tfrac{\sin \Omega t}{\pi t} = \tfrac{\Omega}{\pi} \operatorname{sinc} \tfrac{\Omega}{\pi} t$$

$$\operatorname{sinc} \xi = \frac{\sin \pi \xi}{\pi \xi}$$

Sinc function

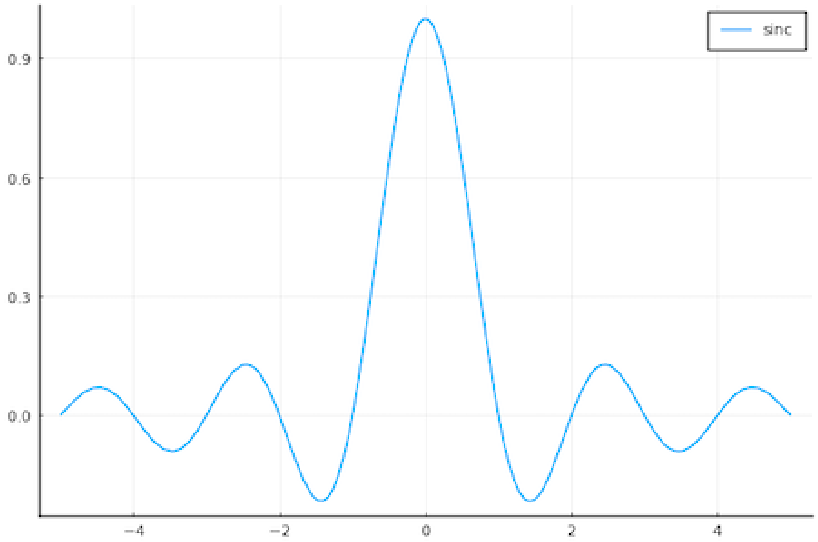


Figure 5: $\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$.

Poisson Summation Formula

- T-periodization of f: $f_T(x) = \sum_{k\in\mathbb{Z}} f(x+kT)$
- Fourier series:

$$\begin{aligned}
c_k(f_T) &= \frac{1}{T}\int_{-T/2}^{T/2} f_T(x) e^{-i\frac{2\pi}{T}} dx \\
&= \frac{1}{T}\int_{-T/2}^{T/2} \sum_{s=-\infty}^{\infty} f(x+sT) e^{-i\frac{2\pi k}{T}x} dx \\
&= \frac{1}{T}\sum_{s=-\infty}^{\infty} \int_{-T/2}^{T/2} f(x+sT) e^{-ik\frac{2\pi}{T}x} dx \\
&= \frac{1}{T}\int_{-\infty}^{\infty} f(x) e^{-ik\frac{2\pi}{T}x} dx = \frac{1}{T}\hat{f}\left(\frac{2\pi k}{T}\right).
\end{aligned}$$

- Therefore: $f_T(x) = \frac{1}{T}\sum_k \hat{f}\left(\frac{2\pi k}{T}\right) e^{i\frac{2\pi}{T}kx}$
- Analogously, if τ is the sampling period then: $\hat{f}_{2\pi/\tau}(\xi) = \sum_{n=-\infty}^{\infty} \hat{f}\left(\xi + \frac{2\pi}{\tau}n\right) = \tau\sum_k f(\tau k) e^{-i\tau k\xi}$

Nyquist sampling theorem

Suppose f is $\frac{\pi}{\tau}$-bandlimited: $\hat{f}(\xi) = \hat{f}_{\frac{2\pi}{\tau}}(\xi)\chi_{[-\frac{\pi}{\tau},\frac{\pi}{\tau}]}$. Then

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)e^{i\xi x}d\xi = \frac{1}{2\pi}\int_{-\pi/\tau}^{\pi/\tau} \hat{f}_{2\pi/\tau}(\xi)e^{i\xi x}d\xi \\
&= \frac{\tau}{2\pi}\int_{-\pi/\tau}^{\pi/\tau} \sum_k f(\tau k)e^{-i\tau k\xi}e^{i\xi x}d\xi \\
&= \frac{\tau}{2\pi}\sum_k f(\tau k)\int_{-\pi/\tau}^{\pi/\tau} e^{i\xi(x-\tau k)}d\xi \\
&= \underline{\sum_k f(\tau k)\,\mathrm{sinc}\,\frac{x-\tau k}{\tau}}
\end{aligned}
$$

Bandlimited functions

supp $\hat{f} \subset [-\Omega, \Omega]$, where $\Omega < \frac{\pi}{\tau}$.

$$f(x) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} e^{i\xi x} \left(\int_{\mathbb{R}} e^{-i\xi t} f(t) dt \right) d\xi$$
$$= \int_{\mathbb{R}} f(t) \frac{\sin \Omega(x-t)}{\pi(x-t)} dt.$$

Ideal low-pass filter:

$$P_\Omega f(x) = \int_{\mathbb{R}} \frac{\sin \Omega(x-t)}{\pi(x-t)} f(t) dt.$$