Lecture 3
Regularization

Topics in Inverse Problems and Super-Resolution
Fall 2022

Recap

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- Singular Value Expansion ("operator SVD")

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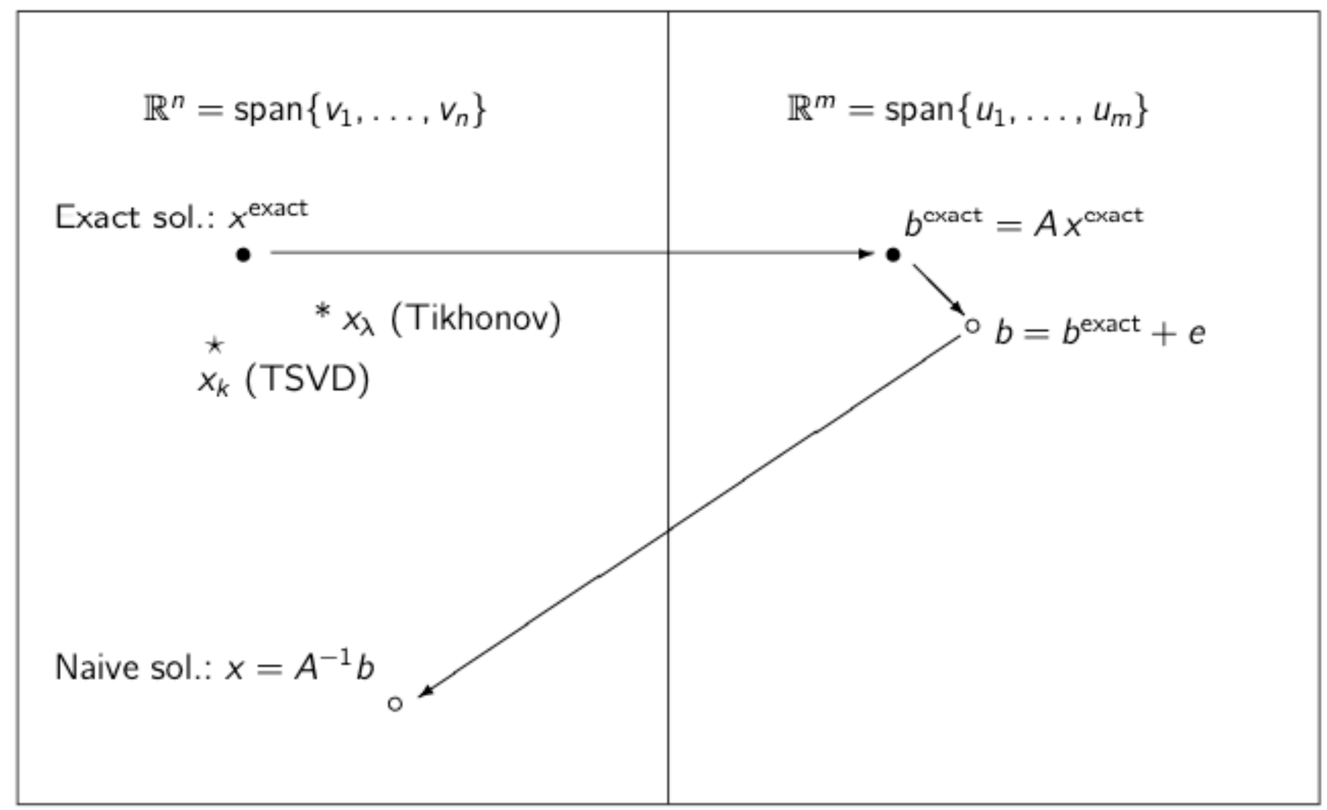

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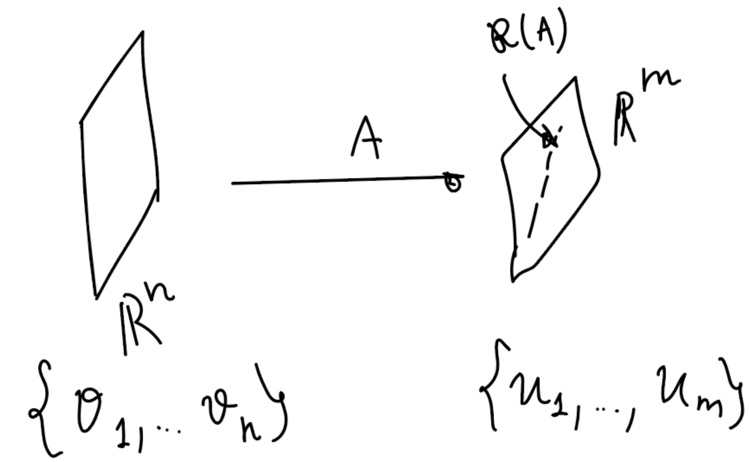

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$$Ax = \sum_{i=1}^{n} \mu_i \langle x, v_i \rangle u_i .$$

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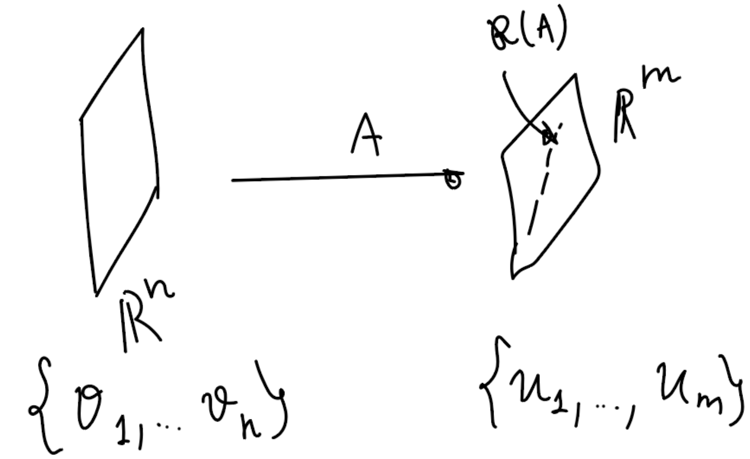

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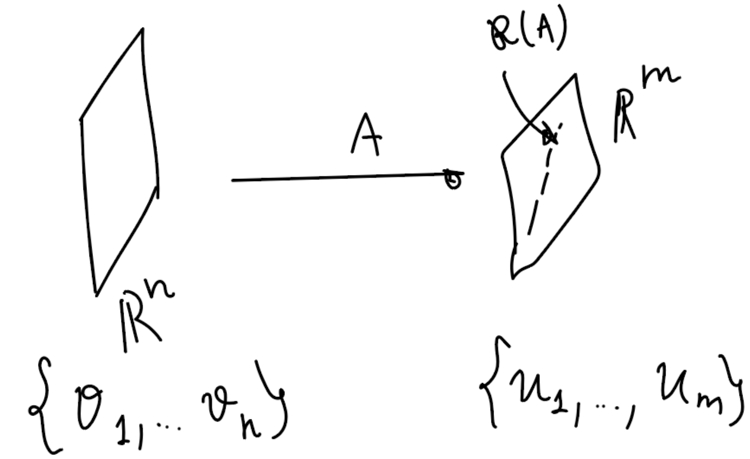

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1. **There exist orthonormal sequences** $\{\varphi_i\} \subset X, \{\psi_i\} \subset Y$ **(right and left singular functions, resp.) such that**

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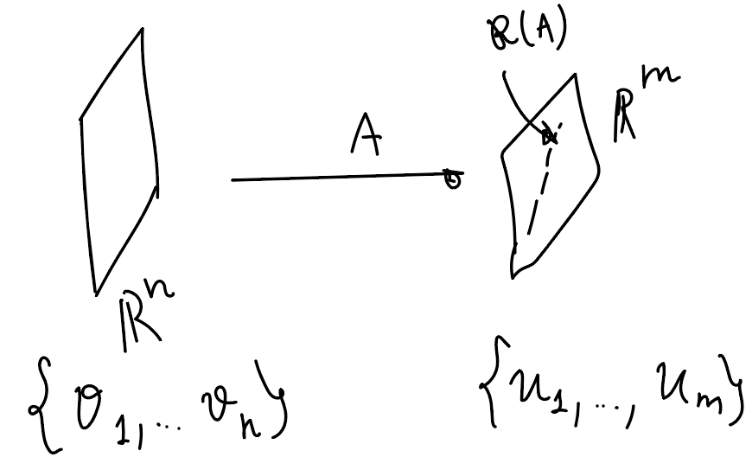

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$$A\varphi_i = \mu_i \psi_i, \quad A^*\psi_i = \mu_i \varphi_i.$$
2. **For each** $\varphi \in X$ **there holds** $\varphi = \sum \left\langle \varphi, \varphi_i \right\rangle \varphi_i + Q\varphi$**, where** Q **is the orthogonal projection onto** $\mathcal{N}(A)$**, and**
$$A\varphi = \sum_{i=1}^{\infty} \mu_i \left\langle \varphi, \varphi_i \right\rangle \psi_i \,.$$

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(Parseval: $\|\varphi\|^2 = \sum_{i=1}^{\infty} \left| \left\langle \varphi, \varphi_i \right\rangle \right|^2 + \|Q\varphi\|^2$**)**

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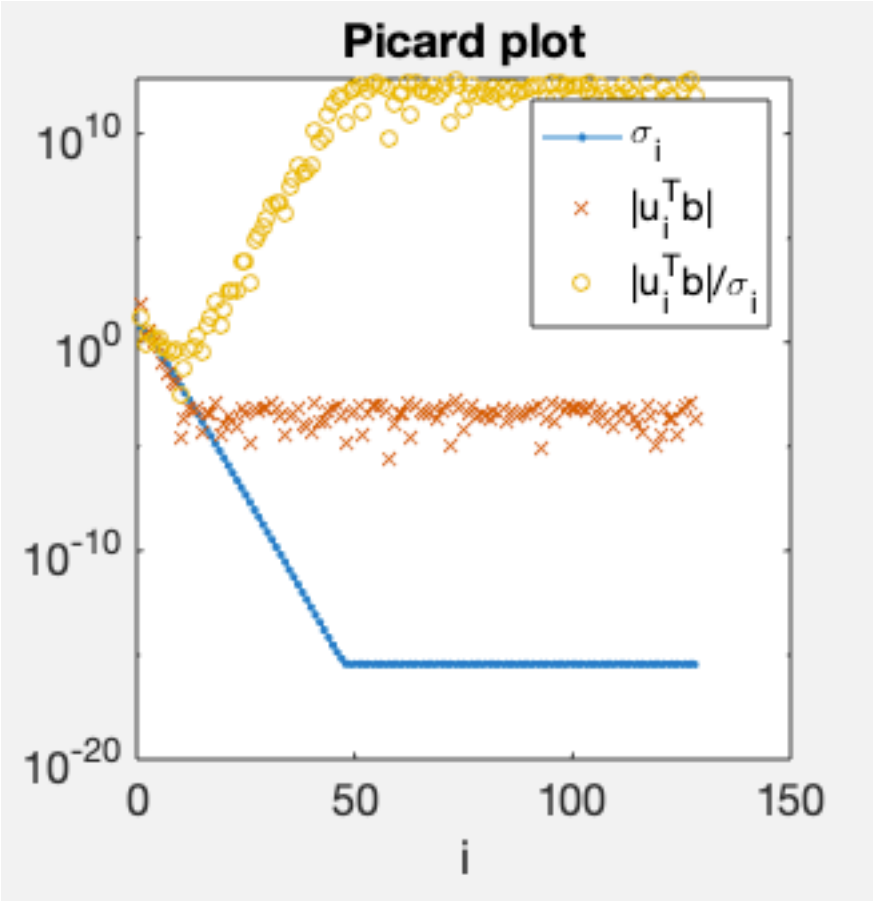

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As $n \to \infty$, $\mu_n \to 0$ **and so this is no good!**

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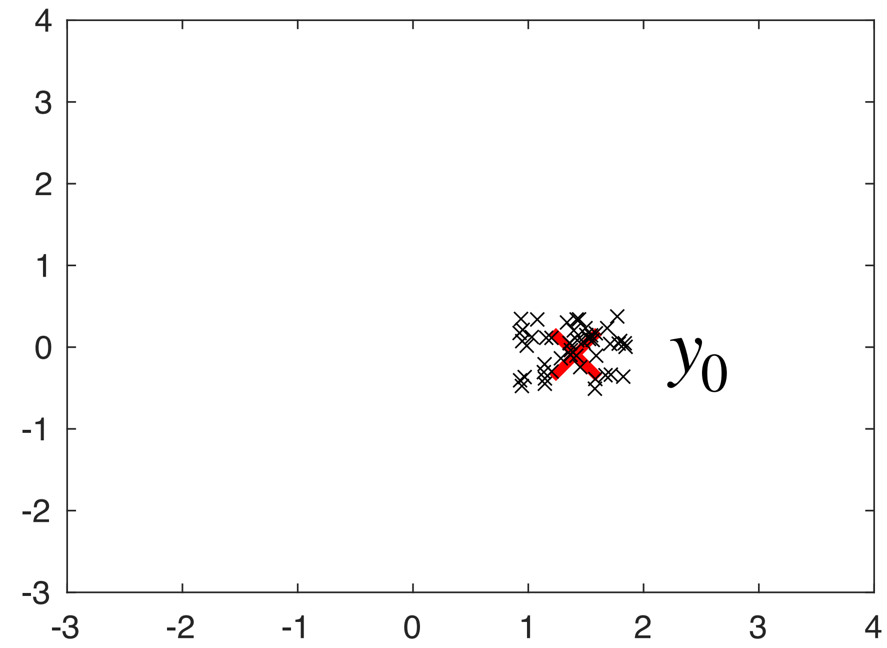

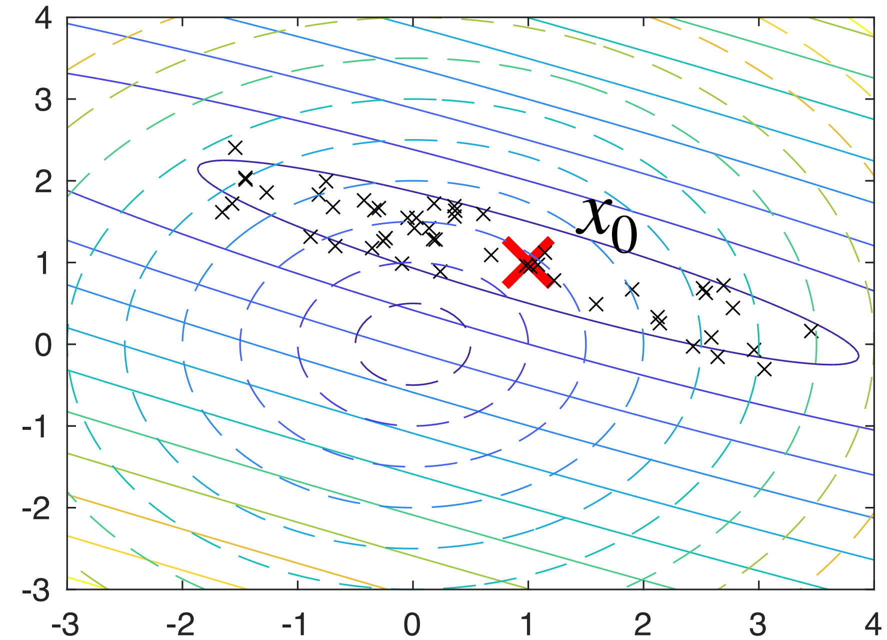

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We assume $y \in \mathscr{R}(A)$

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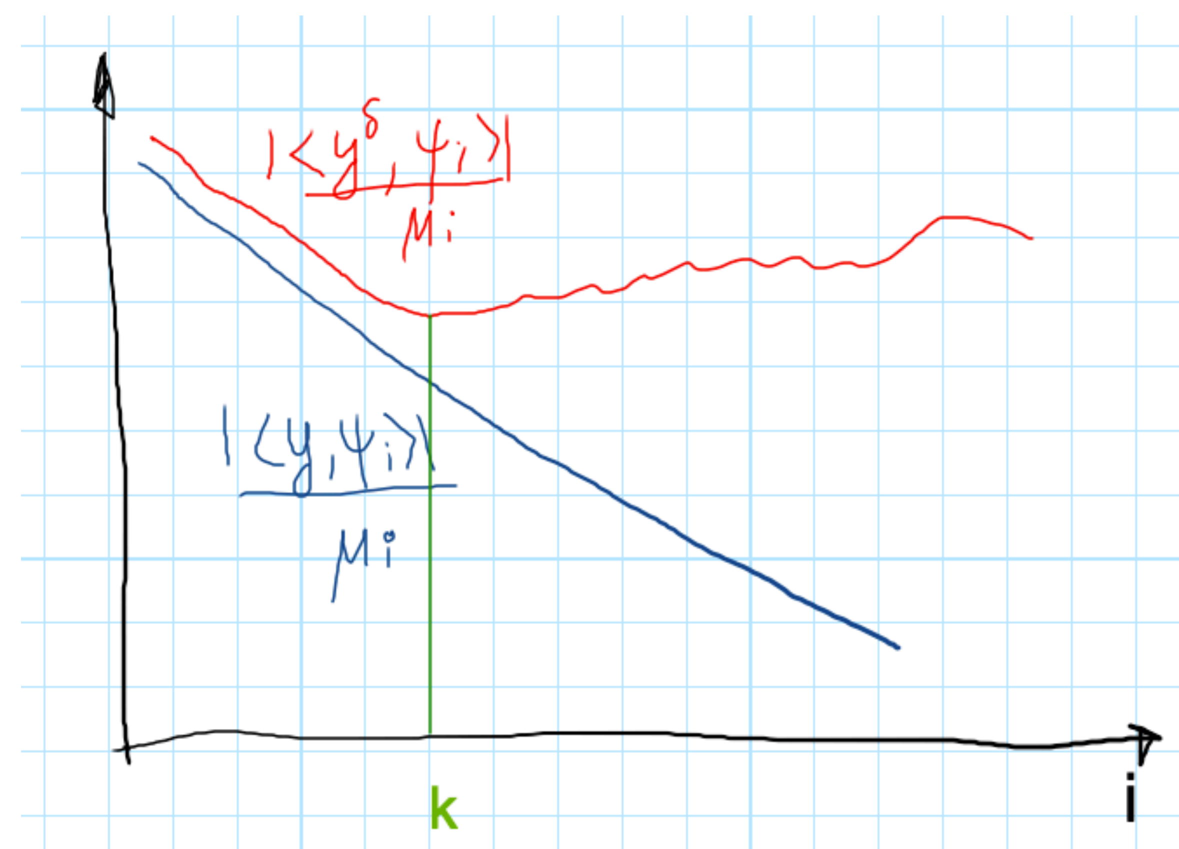
Picard's plot



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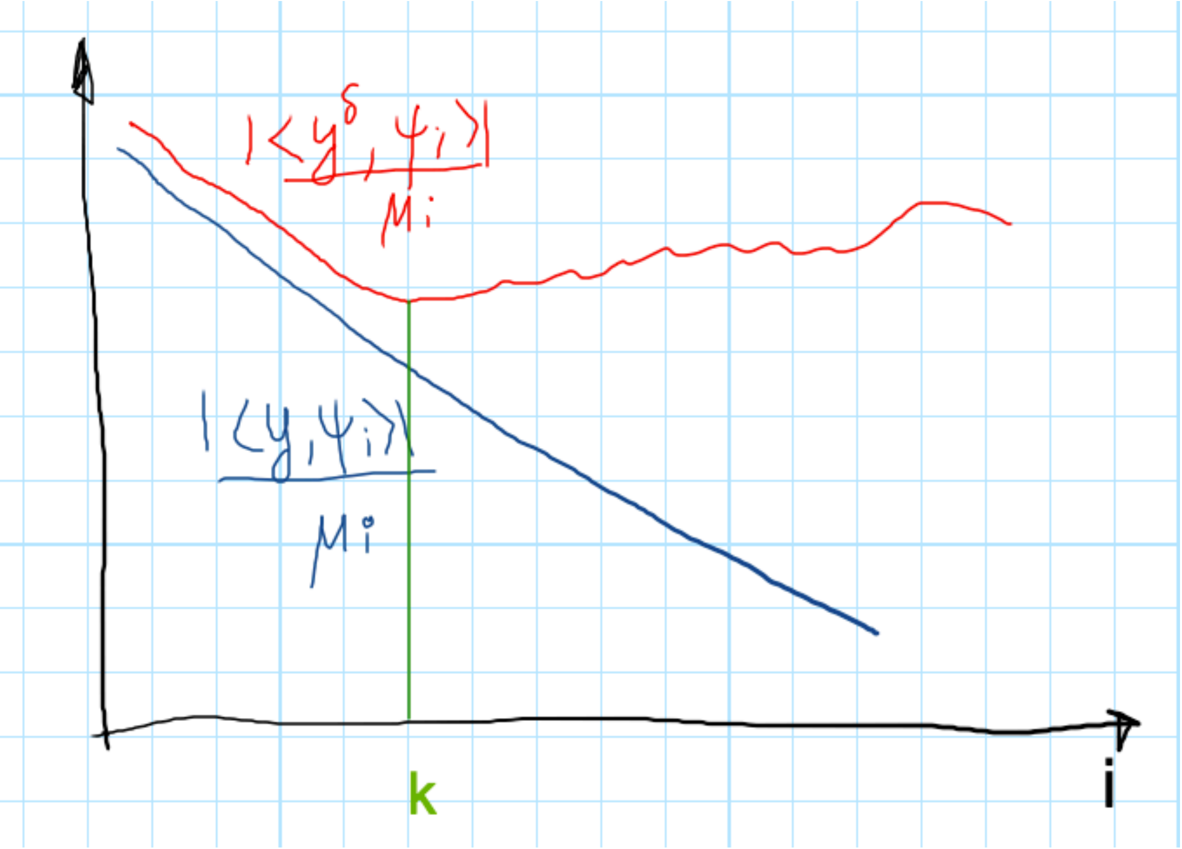

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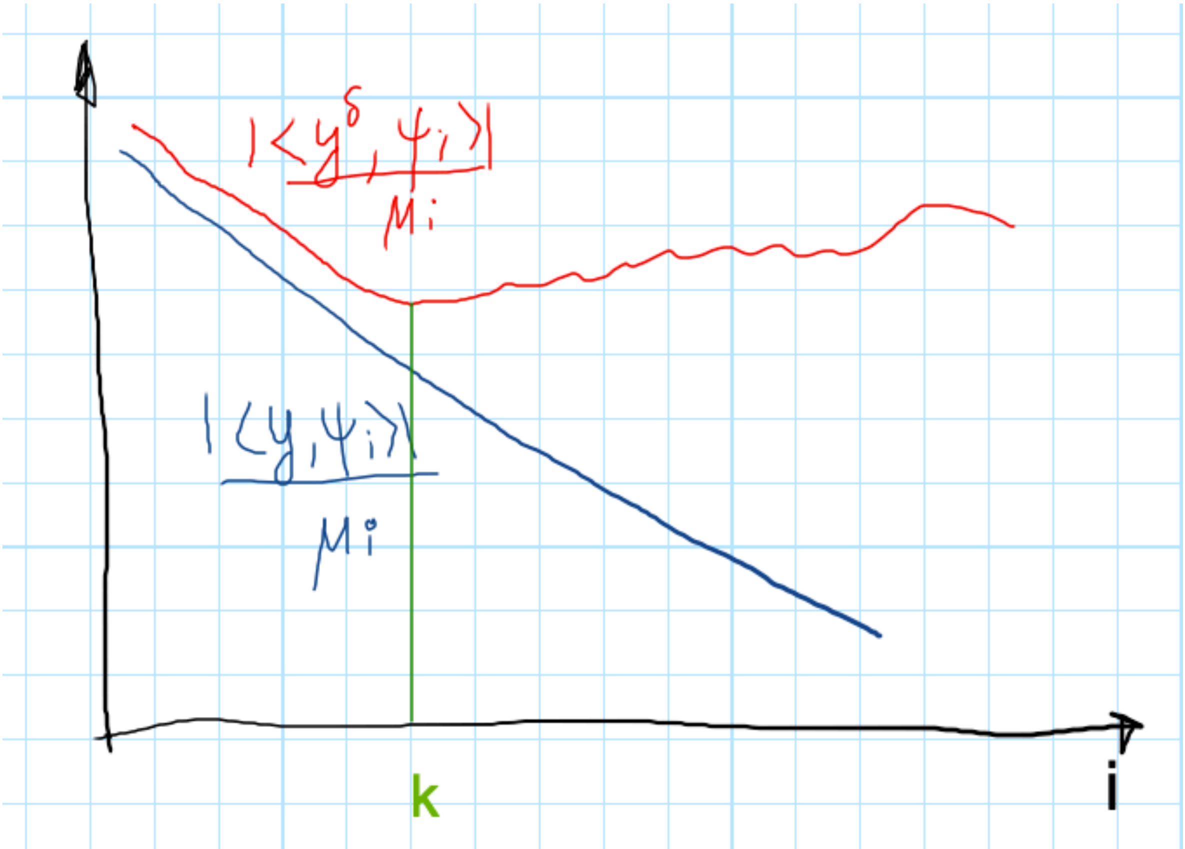

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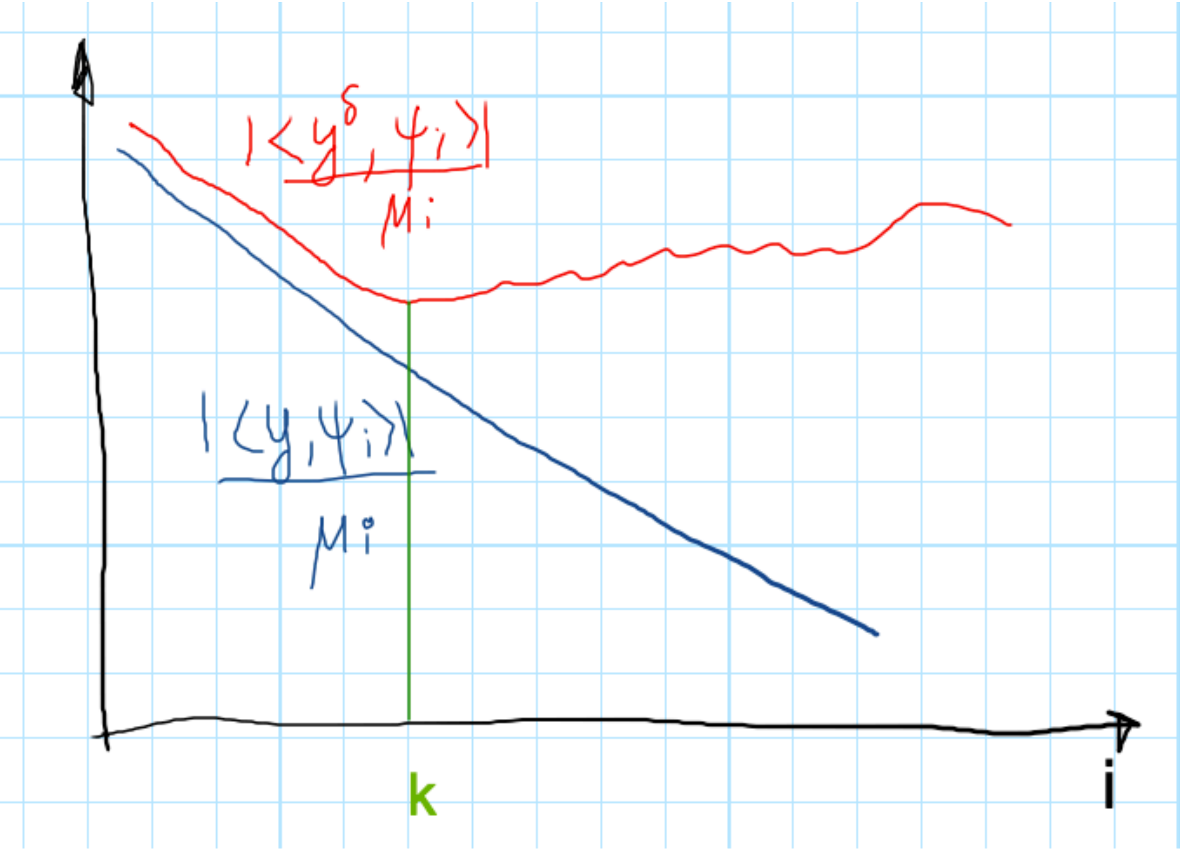

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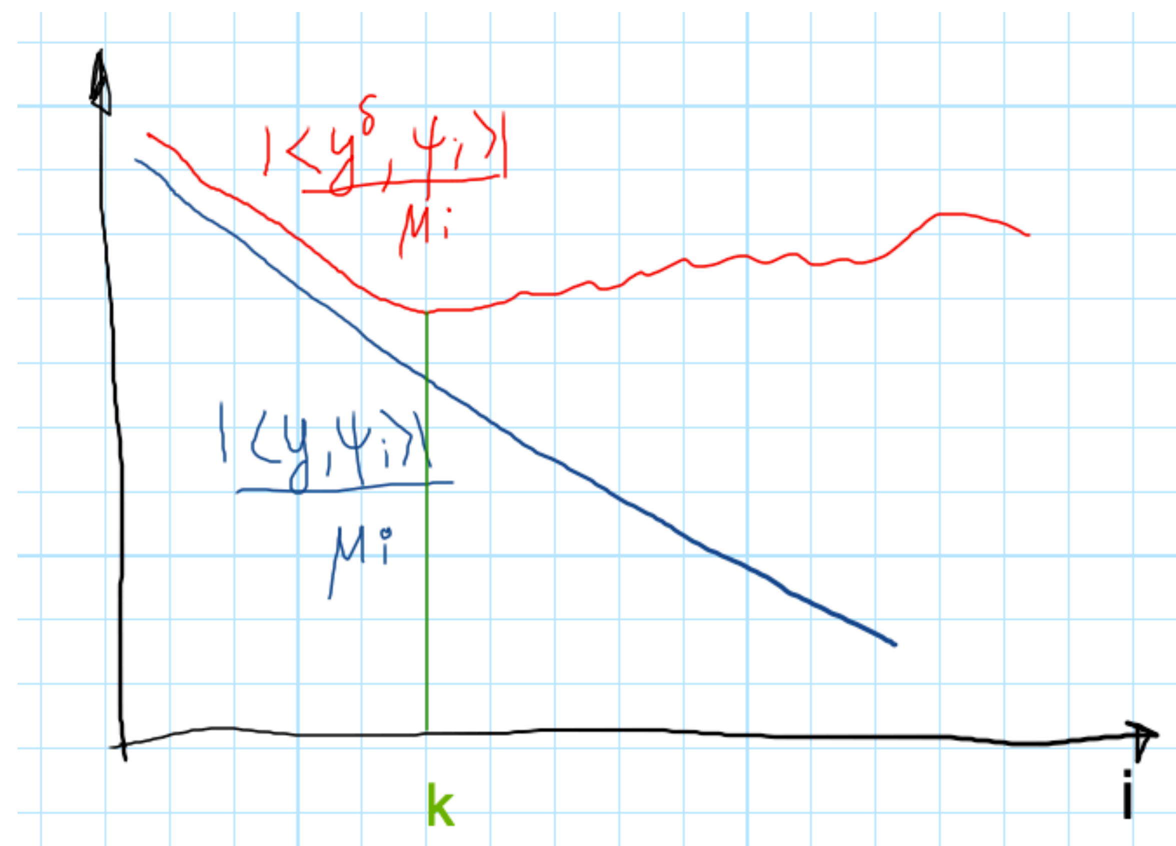

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- How to choose k **optimally?**

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(noise amplification error) **(truncation error)**

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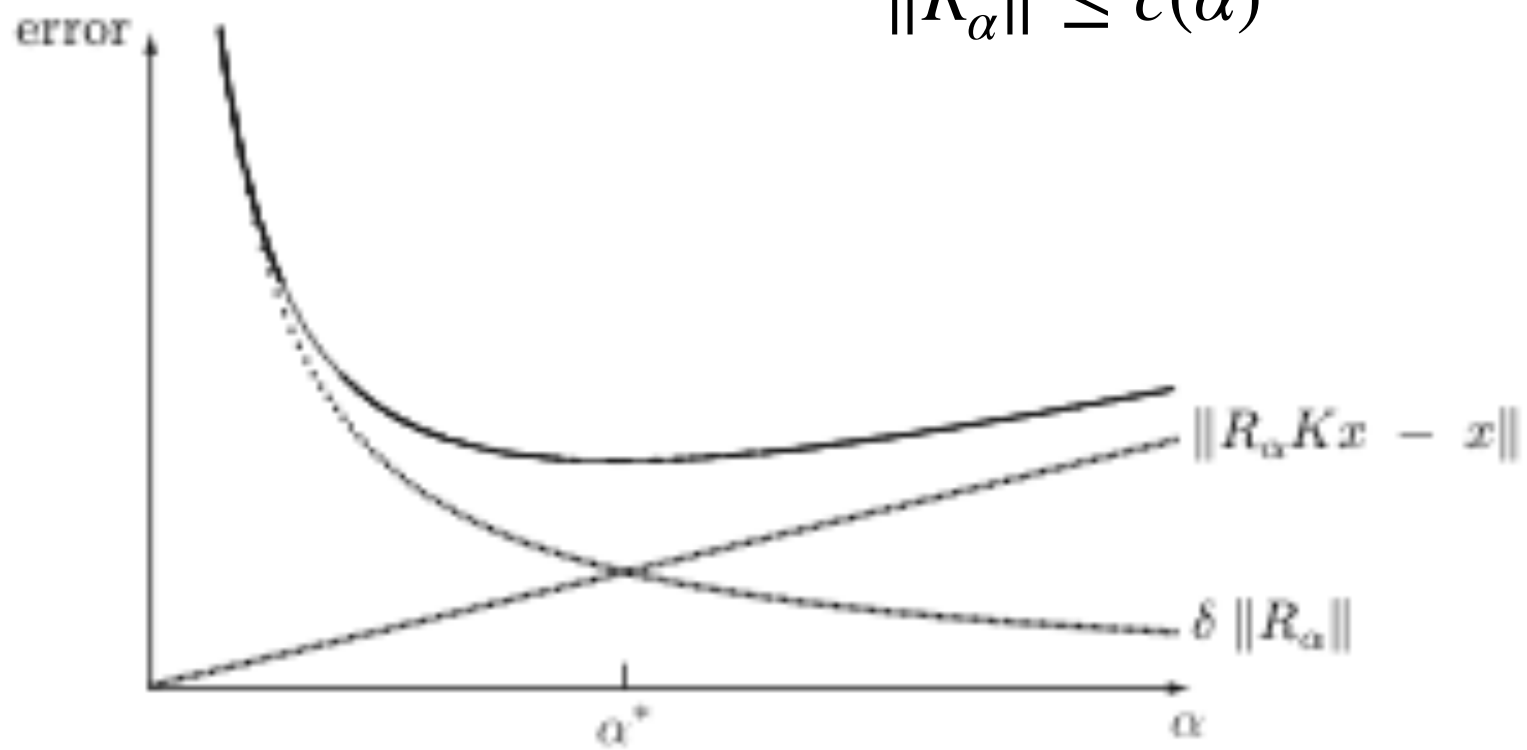

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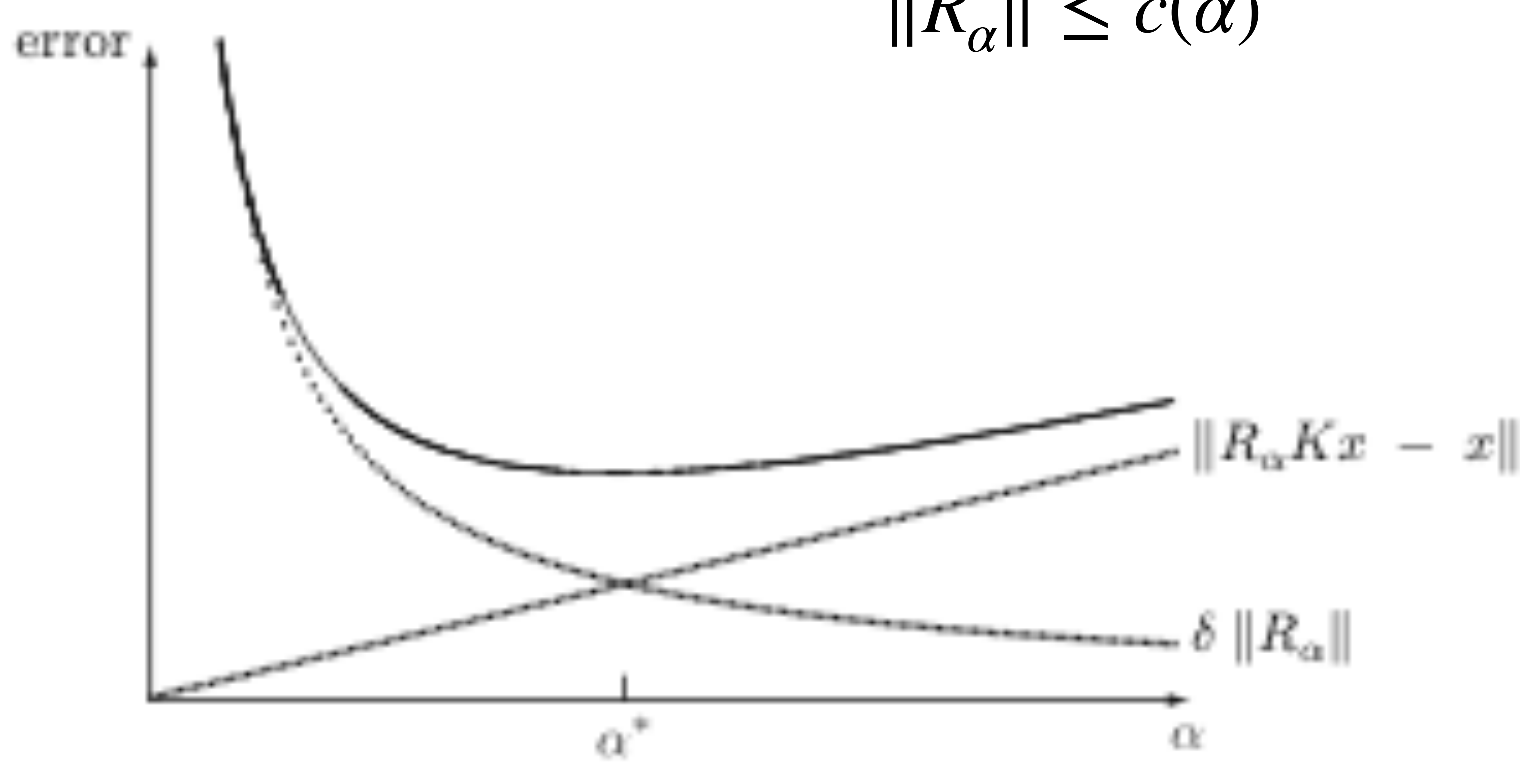

Convergence: choose $\alpha = \alpha(\delta)$ **such that** $\|\delta x\| \to 0$ **as** $\delta \to 0.$

Cntd.

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Next question: rate of convergence

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Is it optimal?

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<u>**Claim:**</u> **assume** $x \in X$ **satisfies** $\|Ax - y^\delta\| \le \delta,\ \|x\|_* \le E$ **for some** $y^\delta \in Y$**.**

Then for any $x_1 \in X$ **satisfying these conditions, we have**

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<u>Note:</u> $\|\cdot\|_*$ **should be "stronger" than** $\|\cdot\|$**, otherwise** $\mathscr{F} \nrightarrow 0$ **as** $\delta \to 0$**.**

$$\|x\| \le c\|x\|_*$$

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Theorem: **Let** $A : X \to Y$ **compact, injective, with dense range.**
Put $\|x\|_* := \|(A^*)^{-1}x\|$**. Then**

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Worst-case error

Definition: $\mathscr{F}(E,\delta,\|\cdot\|_*) = \sup\left\{\|x\| : \quad \|Ax\| \leq \delta,\ \|x\|_* \leq E\right\}.$

Theorem: Let $A : X \to Y$ **compact, injective, with dense range.**
Put $\|x\|_* := \|(A^*)^{-1}x\|$. **Then**

$$\mathscr{F}(E,\delta,\|\cdot\|_*) \leq \sqrt{\delta E}\,.$$

Proof: let $x = A^*z$ **with** $\|z\| \leq E$. **Then**

$$\|x\|^2 = \langle A^*z, x\rangle = \langle z, Ax\rangle \leq \|z\|\,\|Ax\| = \delta E\,.$$

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Optimality: consider $z_n = E\psi_n$.

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- This works for general operators B and filters $q(\alpha, \mu)$ satisfying certain technical conditions (not too complicated…)