Inverse Problems - Lecture 2

0372402501 - Topics in Inverse Problems and Super Resolution, Fall 2022/23

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Main references

- Kress, Rainer, V. Maz'ya, and V. Kozlov. 1989. *Linear Integral Equations*. Vol. 82. Springer.
- Kirsch, Andreas. 2011. *An Introduction to the Mathematical Theory of Inverse Problems*. 2nd ed. Applied Mathematical Sciences, v. 120. New York: Springer.

Well-posed and ill-posed problems

Hadamard's criterion

A (direct/inverse) problem is **well-posed** if

1. it has a solution;
2. the solution is unique;
3. the solution depends **continuously** on the data.

Otherwise, a problem is called **ill-posed**.

Hadamard's example (1923)

Consider Cauchy's problem for the Laplace equation:

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0,\ u(x,0) = 0,\ \frac{\partial u}{\partial y}(x,0) = \frac{1}{n} \sin nx$$

- a unique solution: $u_n(x,y) = \frac{1}{n^2} \sin nx \sinh ny$
- when $n \to \infty$: $u_n(x,0) \to 0$
- if there was continuity wrt $u(x,0)$, we would have $u_n(x,y) \to 0$ for all x, y
- however $u_n(x,y) \to \infty$ for any $y \neq 0$

For many years such examples were considered to be "non-physical".[1] They were also called **"incorrect" or "improperly posed"**.

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[1] *The above does appear in ECG (reconstructing cardiac electrical activity from given body surface electrocardiographic measurements, see e.g. https://www.ncbi.nlm.nih.gov/pmc/articles/PMC6370732/.*

Inverse heat propagation

Forward problem

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t} \\
u(x,0) &= f(x) \quad \text{initial heat distribution} \\
u(0,t) &= u(1,t) = 0, \\
0 &\leq x \leq 1 \\
t &\geq 0,
\end{aligned}
$$

Fourier series

If $f(x) = \sum_{n=1}^{\infty} f_n \sin(\pi n x)$ where $f_n = 2\int_0^1 f(x)\sin(\pi n x)dx$, then by separation of variables

$$u(x,t) = \sum_{n=1}^{\infty} f_n e^{-(\pi n)^2 t} \sin(\pi n x)$$

Smoothing/loss of information

$$f(x) = \sum_{n=1}^{\infty} f_n \sin(\pi n x),\, f_n = 2\int_0^1 f(x)\sin(\pi n x)dx$$

$$u(x,t) = \sum_{n=1}^{\infty} f_n e^{-(\pi n)^2 t}\sin(\pi n x)$$

- Finite precision (say up to ϵ) computations → only $N_{\epsilon,0}$ first coefficients of f will be nonzero
- For $t = T$ (final time) only $N_{\epsilon,T} \ll N_{\epsilon,0}$ nonzero coefficients in the data $u(x,T)$

Ill-posedness of the inverse

- Now if $u_n(x,T) = \frac{1}{n}\sin(\pi n x)$ then $u_n(x,0) = \frac{1}{n}\sin(\pi n x)e^{(\pi n)^2 T}$.
- Again, when $n \to \infty$ we have $u_n(x,T) \to 0$ but $u_n(x,0) \to \infty$.
- Well-posed if e.g. $f_n \sim e^{-n^3}$ (not expected to occur in practice)
- The continuity of the inverse depends on the **function spaces/norms** chosen.

Integral equations

Recall $f_n = 2\int_0^1 u(y,0)\sin(\pi n y)dy$. Consider

$$K_t(x,y) := 2\sum_n e^{-(\pi n)^2 t}\sin(\pi n x)\sin(\pi n y) \quad \text{(the heat kernel)}$$

(also known as the **Green function** for the heat equation)

Then can write

$$u(x,t) = \sum_n f_n \sin(\pi n x) e^{-(\pi n)^2 t} = \int_0^1 u(y,0) K_t(y,x) dy$$

In general:

- Direct problem: given K, f, compute g (“solve” the PDE)

$$g(t) = \int_0^1 K(s,t) f(s) ds, \quad 0 \le t \le 1.$$

- Inverse problem: given K, g, find f

Operator equations

$$Ax = y, \quad x \in X,\ y \in Y$$

- X, Y - some function spaces (equipped with appropriate norms)
- When A is a **compact** integral operator, it behaves much like an infinite matrix with diagonal elements $\to 0$.

Hilbert space setting

- X is an (∞-dim.) Hilbert space

$$X := \left\{ f : [0,1] \to \mathbb{R}(\mathbb{C}) : \int_0^1 |f|^2 < \infty \right\} \equiv L^2[0,1].$$

- Inner product: $\langle f, g \rangle = \int f \bar{g}$
- Norm: $\|f\| = \sqrt{\langle f, f \rangle} \geq 0$, $\|f\| = 0 \iff f = 0$ (almost everywhere, a.e.)
- Completeness: every Cauchy sequence f_n converges to $f \in L^2[0,1]$

Orthonormal bases

We assume there exists an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ of X (the space is *separable*).

$$\langle \varphi_k, \varphi_\ell \rangle = \delta_{k,\ell} = \begin{cases} 1 & k = \ell \\ 0 & \text{else} \end{cases}$$

Examples: Fourier basis $\{e^{2\pi nx}\}$, orthogonal polynomials, spherical harmonics,. . .

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Non-separable example

$X := \left\{ f = \sum_{k=1}^{n} c_k e^{\imath \lambda_k t}, \quad \lambda_k \in \mathbb{R} \right\}$ (finite linear combinations)
Inner product:

$$\langle f, g \rangle := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t) \overline{g(t)} dt$$

See [Stein & Skarachi, *Real Analysis*, Ch. 4, Problem 2]

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[Another non-separable (Banach space) example: $L^{\infty}([0,1])$]

Bounded operators

- X, Y Hilbert spaces (or even Banach spaces)
- $A : X \to Y$ **bounded** if $\exists C > 0$ s.t. $\|Ax\|_Y \leq C\|x\|_X$ for $\forall x \in X$.
- Operator norm: $\|A\| := \sup_{\|x\|\leq 1} \|Ax\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|}$.
- Unbounded (forward) operators: much more difficult to treat (but e.g. quantum mechanics).

Linear operators

In this part of the course we will deal with linear operators $A : X \to Y$ only.

Many interesting inverse problems are linear, but even more are nonlinear. We will see some examples later.

Continuous operators

A is **continuous** if whenever $x \to x_n$ (i.e. $\|x - x_n\| \to 0$, not pointwise), then $Ax_n \to Ax$.

Lemma 1. Let A be a linear operator. A is continuous iff A is continuous at $x = 0$.

Lemma 2. A linear op. is bounded iff it is continuous.

Proof:

- $\Longrightarrow$: A bounded $\to \|Ax_n\| \leq C\|x_n\| \to 0$ (since $\|x_n\| \to 0$)
- $\Longleftarrow$: Suppose A is unbounded, then $\exists\{x_n\}$ with $\|x_n\| = 1$ but $\|Ax_n\| \geq n$. Define $y_n := \frac{x_n}{\|Ax_n\|}$, then $\|y_n\| \to 0$ but $\|Ay_n\| = 1$, for all n.

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Identity is not compact

Consider $\{\varphi_n\}_{n=1}^{\infty}$ an orthonormal basis (i.e. $\dim(X) = \infty$)

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Identity is not compact

Consider $\{\varphi_n\}_{n=1}^{\infty}$ an orthonormal basis (i.e. $\dim(X) = \infty$)

- Bounded - OK
- $\{\mathrm{Id}\,\varphi_n\}$ does not contain a converging subsequence:

$$\|\varphi_m - \varphi_n\|^2 = \|\varphi_n\|^2 + \|\varphi_m\|^2 = 2$$

Inverse of compact

Proposition

$A : X \to Y$ compact, $B : Y \to Z$ bounded $\implies BA : X \to Z$ is compact.

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Corollary

If $A : X \to Y$ is compact, then A^{-1} (if exists), cannot be continuous (bounded).

- In this case, the inverse problem is ill-posed.

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Proof: Exercise

Adjoint operators

$$A : X \to Y, \quad X, Y \text{ Hilbert spaces.}$$

A is bounded $\Longrightarrow \exists! A^* : Y \to X$ s.t.

$$\forall x \in X, y \in Y : \quad \langle Ax, y \rangle_Y = \langle x, A^* y \rangle_X$$

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Finite rank operators

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- Example: projection onto a finite number of coordinates

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Proposition: if $\{A_n\}$ are compact and $A_n \to A$ (in the operator norm), then A is compact.

Compact integral operators

Lemma 3: Let $K(s,t) \in L^2([0,1] \times [0,1])$, K continuous. Then

$$\begin{aligned} A : L^2([a,b]) &\to L^2([c,d]) \\ f(s) &\mapsto \int_a^b K(s,t) f(s) ds, \quad t \in [c,d] \end{aligned}$$

is a (bounded) **compact** operator. Furthermore,

$$A^* g = \int_c^d K(s,t) g(t) dt, \quad s \in [a,b].$$

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is a (bounded) **compact** operator. Furthermore,

$$A^* g = \int_c^d K(s,t) g(t) dt, \quad s \in [a,b].$$

Therefore, the inverse problem is always ill-posed.

Proof:

- Let $\{\varphi_n\} \subset X$ and $\{\psi_n\} \subset Y$ be orthonormal bases for $X = L^2([a, b])$ and $Y = L^2([c, d])$.

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- Since $K \in L^2(X \times Y)$ we can write

$$K(s,t) = \sum_{k,\ell} c_{k,\ell}\varphi_k(s)\psi_\ell(t), \quad \sum_{k,\ell} |c_{k,\ell}|^2 < \infty.$$

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- Define

$$K_n(s, t) := \sum_{k=1}^{n} \sum_{\ell=1}^{n} c_{k,\ell}\varphi_k(s)\psi_\ell(t)$$

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- $A_n \to A$, A_n is of finite rank (why?)

Operator SVD

Reminder

$A \in \mathbb{R}^{n \times n}$ is a **normal** matrix $(A^* A = AA^*)$ iff it has a complete set of orthonormal eigenvectors $\{v_j\}_{j=1}^{n}$:

$$Av_j = \lambda_j v_j.$$

Matrix SVD

$A \in \mathbb{R}^{m\times n}$ $(m \geq n)$, $A : \mathbb{R}^n \to \mathbb{R}^m$, $\mathrm{rank}(A) = n$.

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- The SVD:

$$Ax = \sum_{i=1}^{n} \mu_i \langle x, v_i\rangle\, u_i$$

$$A = U \operatorname{diag}\{\mu_1, \ldots, \mu_n, 0, \ldots, 0\}\, V^*$$

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Theorem: $\exists \{\varphi_n\}_{n=1}^{\infty}$ orthonormal basis for X and a sequence $\lambda_n \to 0$ s.t.

$$A\varphi_n = \lambda_n \varphi_n, \quad n = 1, 2, \dots$$

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- A has a non-trivial kernel iff $0 \in \mathrm{sp}(A)$

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Theorem: $\exists \{\varphi_n\}_{n=1}^{\infty}$ orthonormal basis for X and a sequence $\lambda_n \to 0$ s.t.

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- All eigenspaces $E_\lambda := \{x \in X : Ax = \lambda x\}$ for $\lambda \neq 0$ are finite-dimensional.
- $\{\lambda_n\}$ is called the spectrum of A.
- A has a non-trivial kernel iff $0 \in \mathrm{sp}(A)$
- the index n can be thought of as “frequency”

Operator SVD/SVE

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$$\begin{aligned} x &= \sum_{n=1}^{\infty} \langle x, \varphi_n \rangle \varphi_n + Qx, \\ Ax &= \sum_{n=1}^{\infty} \mu_n \langle x, \varphi_n \rangle \psi_n. \end{aligned}$$

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Proof outline: take $\{\varphi_n\}$ to be the orthonormal basis of eigenfunctions (with $\lambda_n \neq 0$), and put $\psi_n := \frac{1}{\mu_n} A\varphi_n$.