Topics in Inverse Problems and Super-Resolution

Instructor: Dr. Dmitry Batenkov, dbatenkov@tauex.tau.ac.il

Time&Place: Wednesdays 10-13, Dan David 204

Moodle: https://moodle.tau.ac.il/course/view.php?id=372402501

Reception hours: by appointment / after class

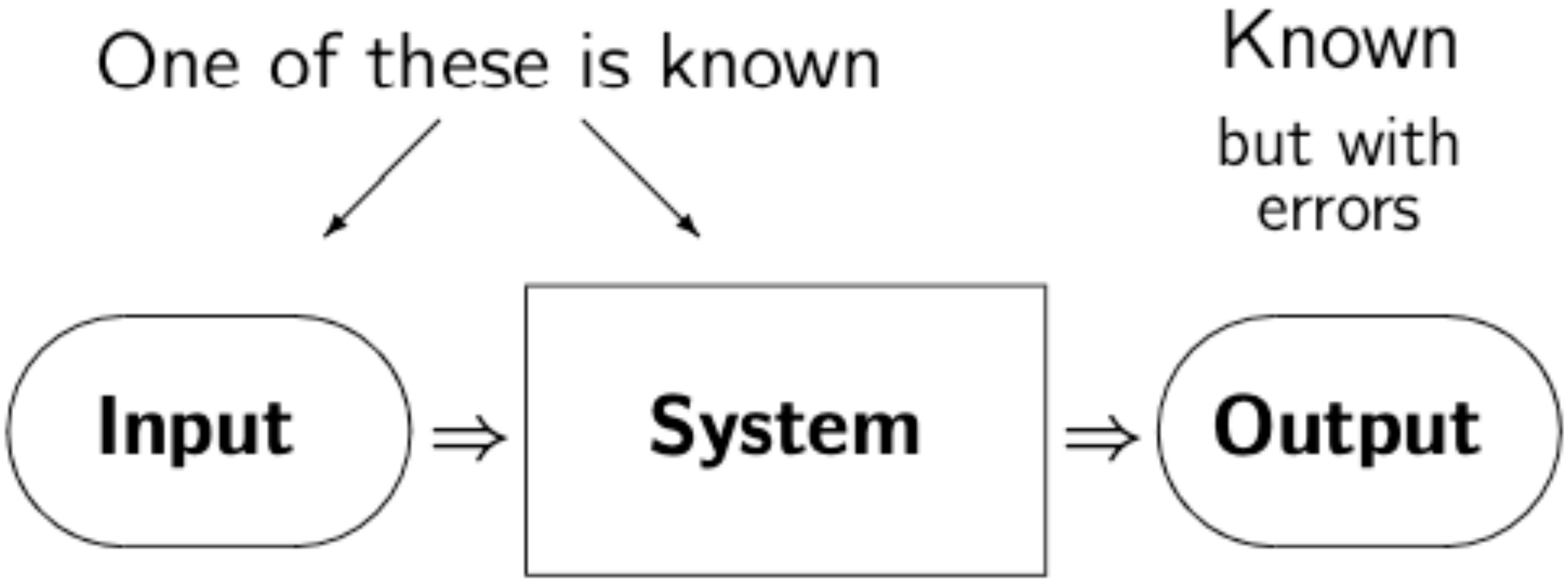
What is an inverse problem?

- Two problems are inverses of one another if the formulation of each involves all or part of the solution of the other.
- Often, for historical reasons, one of the two problems has been studied extensively for some time, while the other has never been studied and is not so well understood.
- In such cases, the former is called the direct problem, while the latter is the inverse problem.

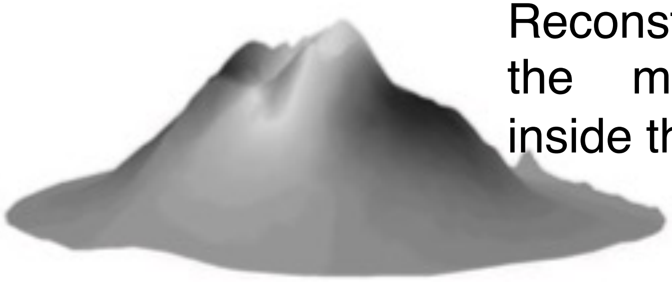
(Keller J B 1976 Am. Math. Monthly 83 107)

For a Mathematician: choice is arbitrary
For a Physicist: direct problem follows **cause→effect**.

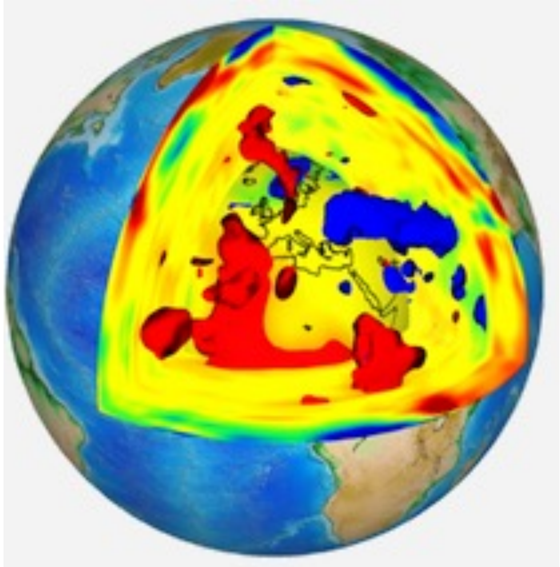
Inverse Problem



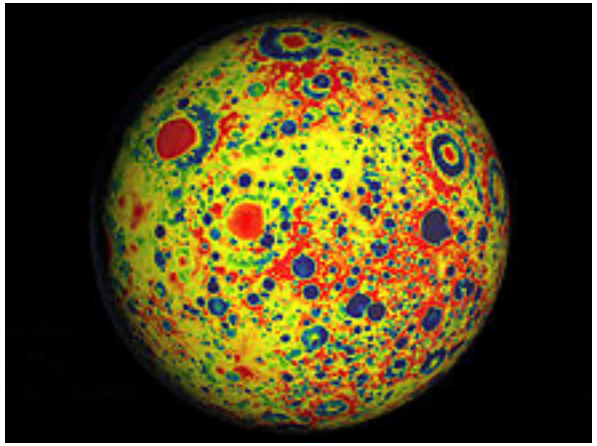
Geophysical inversion



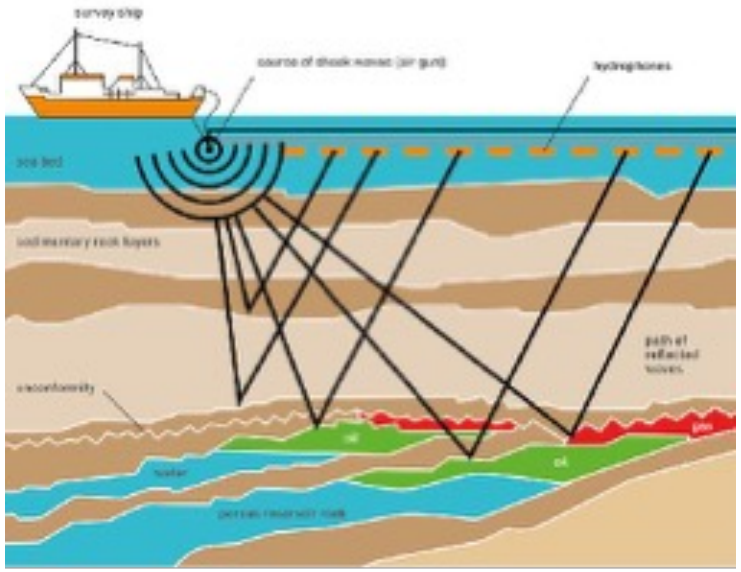
Reconstruction of the magnetization inside the volcano.



Seismic tomography



Gravity map of the Moon
(GRAIL mission)



Seismic inversion

Diffraction-limited imaging

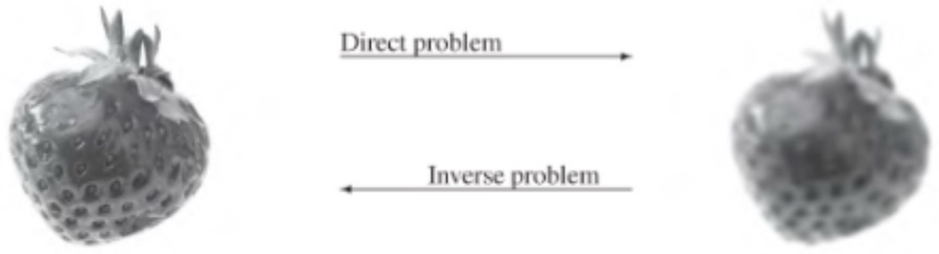
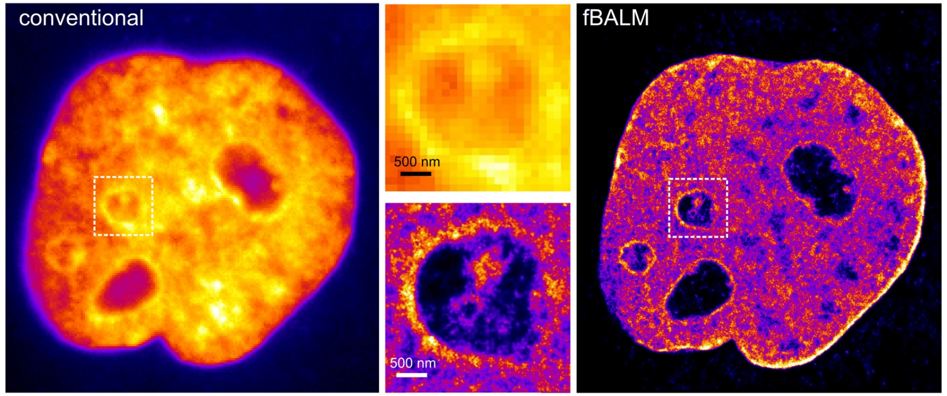
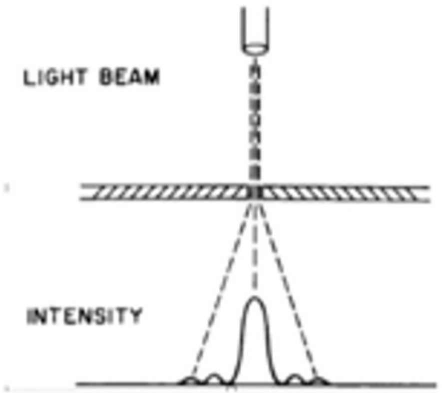
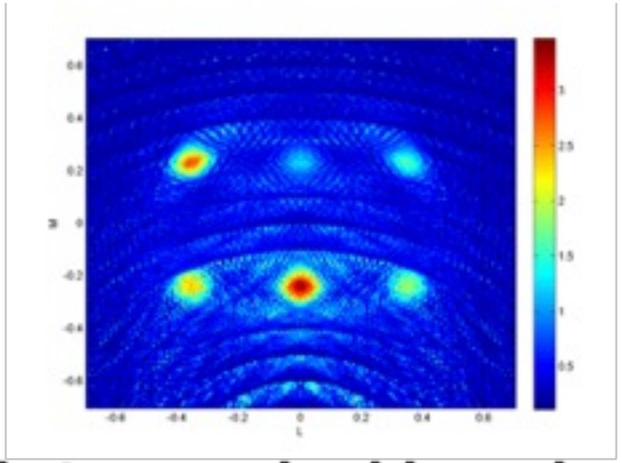


Image deblurring

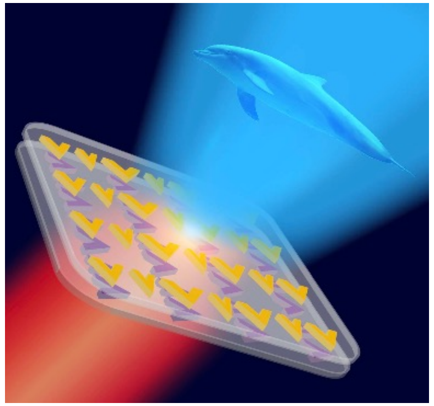


Super-resolution imaging



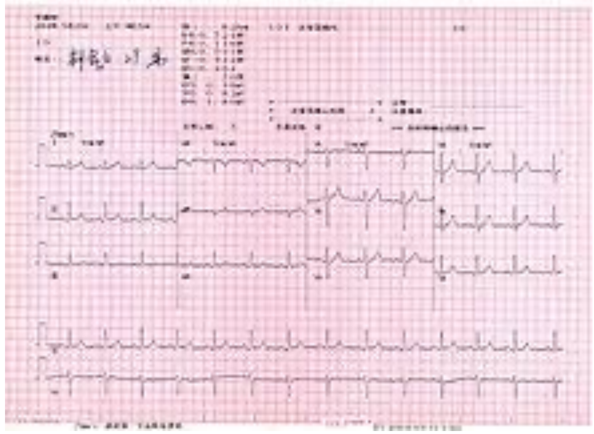


Astronomical imaging

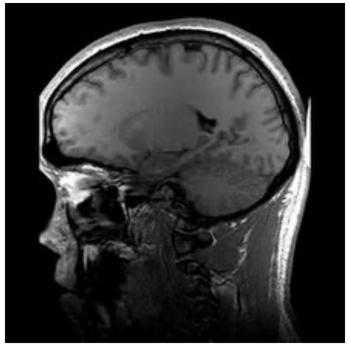


holography

Medical imaging

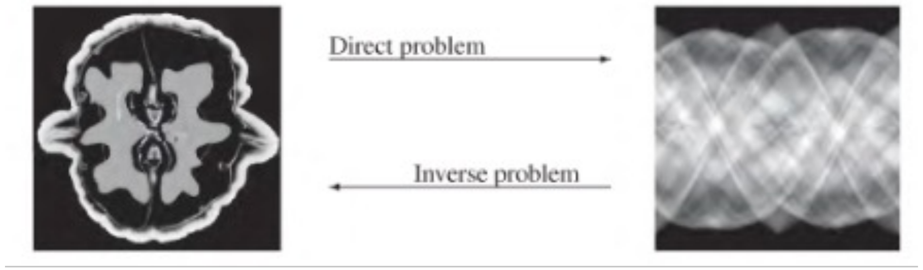


ECG



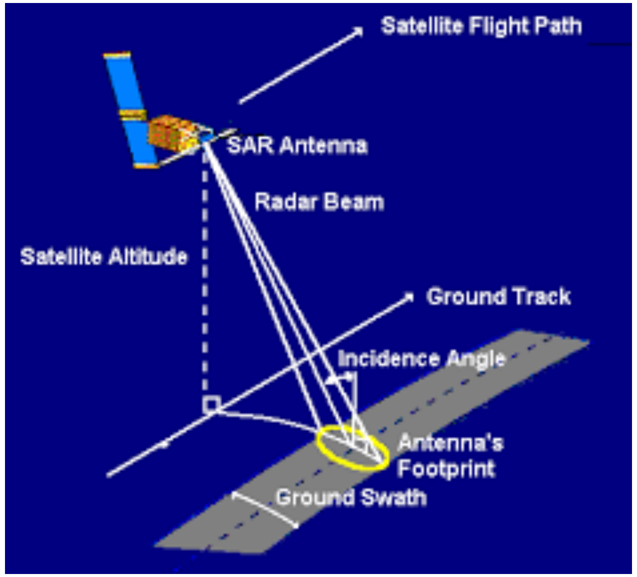
MRI

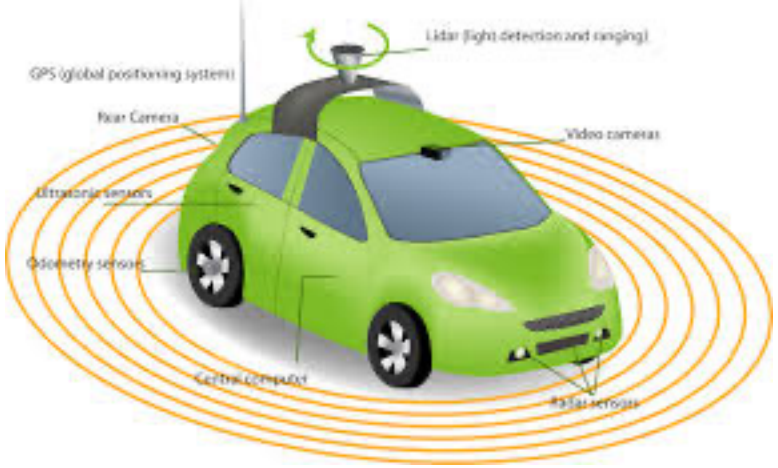
Ultrasound

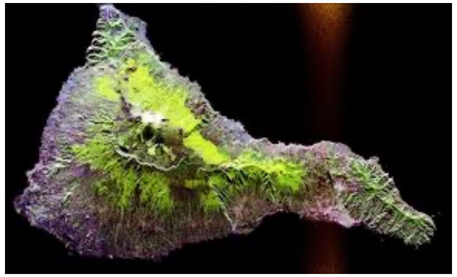


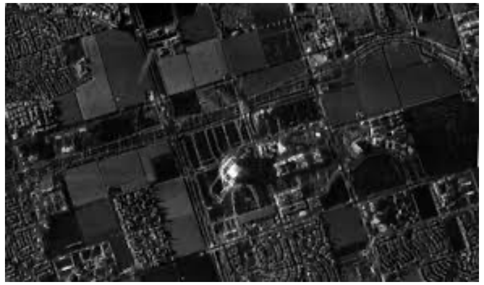
Computerized Tomography

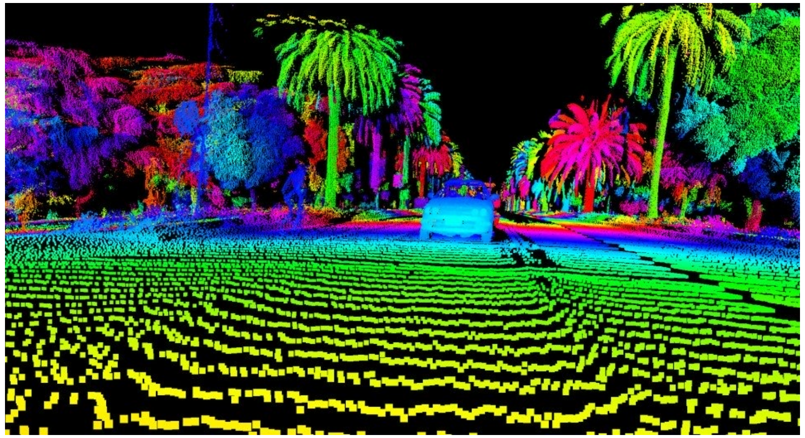
Remote sensing







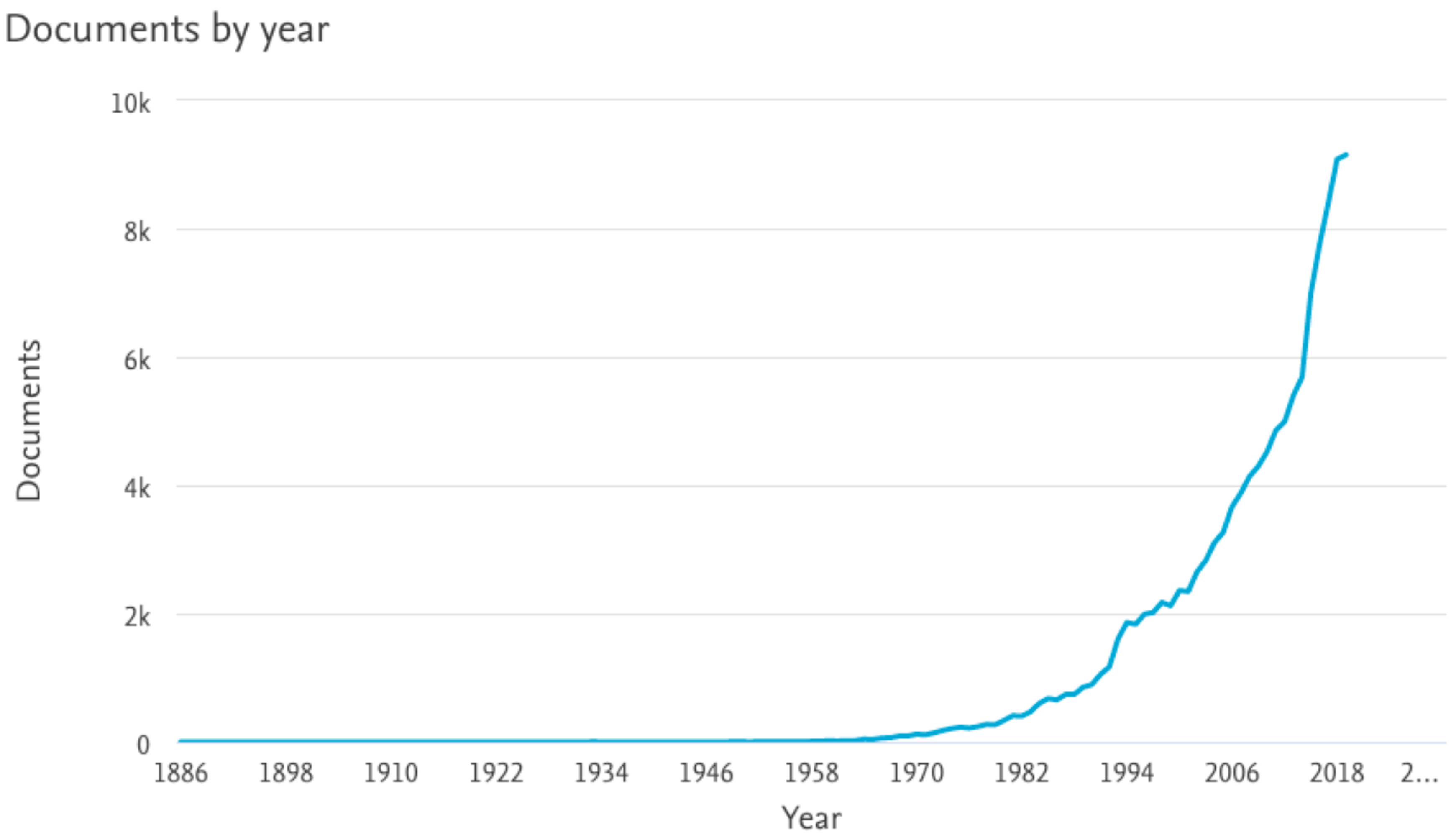




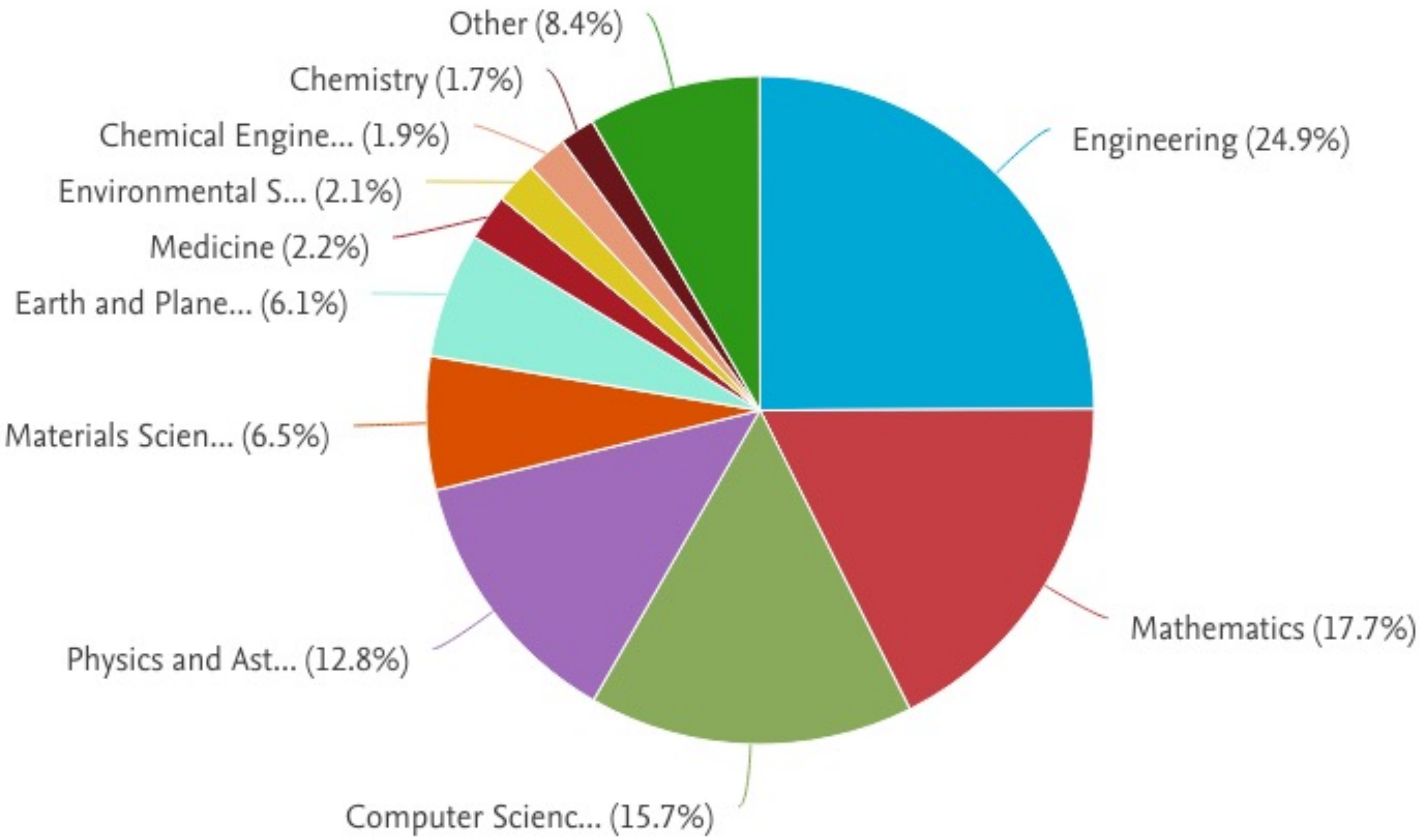
Past, present and future (very inaccurate)

- 19th century: differential equations (forward problems)
- 20th century: mathematical physics, many theoretical developments, “classical” results
- 2000-2010 : sparsity, compressed sensing
- Today: large scale computations, deep learning, “data science”

Inverse Problems - today



By subject area



Math/computational questions

$$y = K(x) + e$$

Modeling!!

- Existence/uniqueness (sampling)
- How to solve efficiently?

$$\text{minimize}_x \|y - K(x)\| + \underbrace{\mathcal{R}(x)}_{\text{prior knowledge}}$$

- Uncertainty quantification / Bayesian IP

$$x \sim p(\theta) \Rightarrow \text{compute } p(x|y)$$

- **Stability/resolution**

$$\|\tilde{x} - x\| \leqslant \kappa(\|e\|), \ \kappa = ?$$

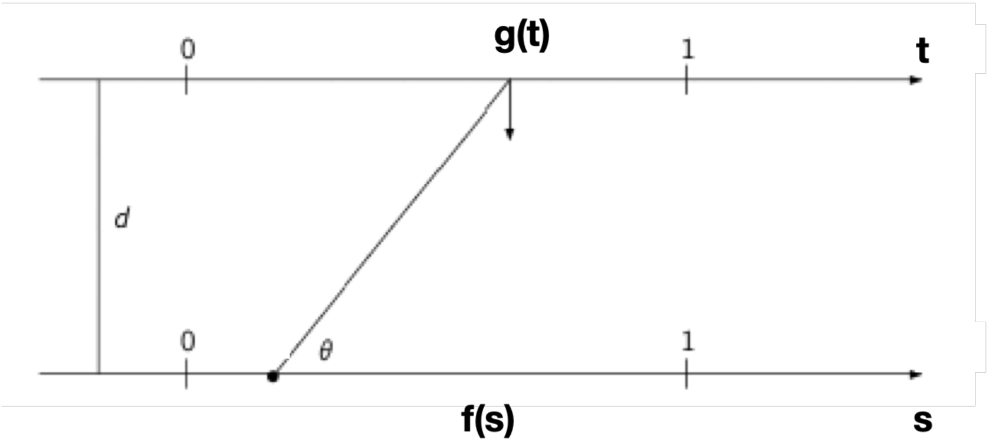
Course topics

1. General theory
2. Regularized solutions
3. Discretization
4. Resolution and super-resolution
5. Sparsity constraints
6. Applications / current research

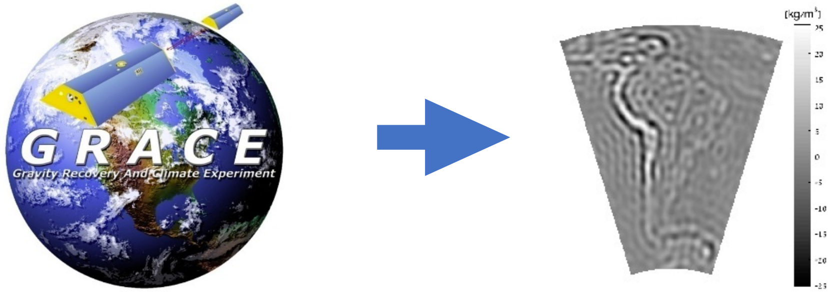
Linear integral equations

$$y(t) = \int K(s,t)x(s)ds + e(t)$$

- Accurate model for many IPs



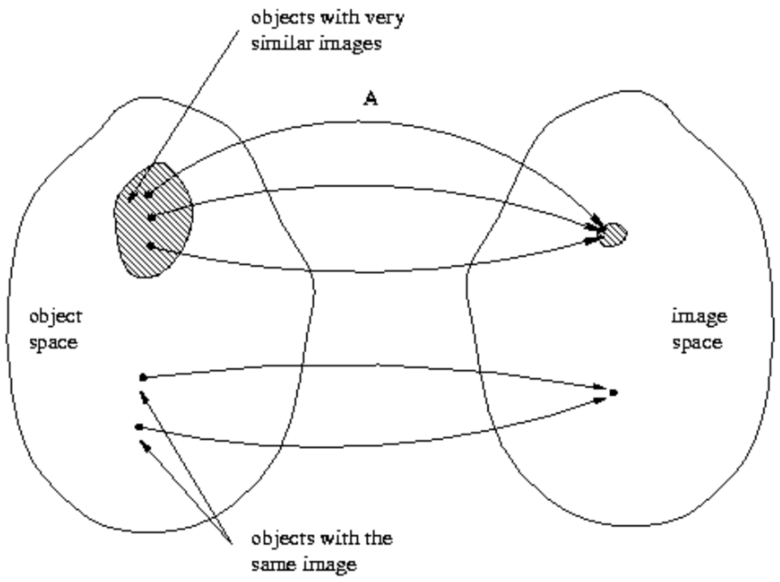
$$g(t) = \int_0^1 \frac{d}{(d^2 + (t-s)^2)^{3/2}} f(s)\mathrm{d}s$$

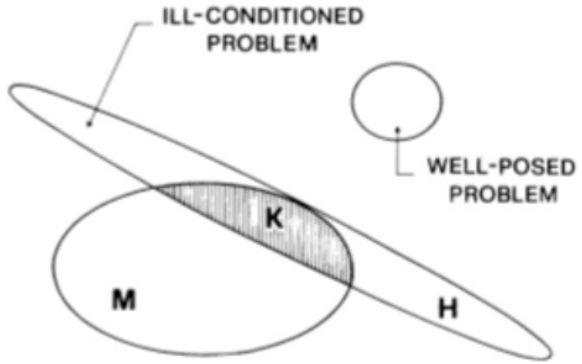


$$(T\rho)(\vec{y}) = V(\vec{y}) = -G\int_{\oplus} \frac{\rho(\vec{x})}{|\vec{x}-\vec{y}|}\mathrm{d}^3x$$

$$y(t) = \int K(s,t)x(s)ds + e(t)$$

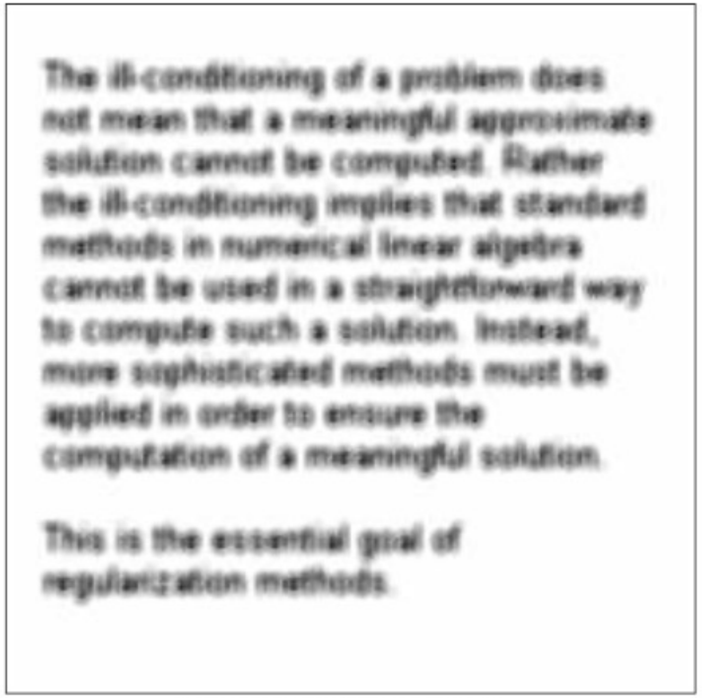
- Singular value expansion of compact operators
- Problem <u>always</u> ill-posed: need prior information in order to solve
  - $||x|| \leq E$ (size/energy)
  - $x(t) \sim \sum_k c_k \varphi_k(t) \ with\ c_k \to 0$ sufficiently fast (e.g. smoothness)
- Worst case error $||x - \tilde{x}||$?



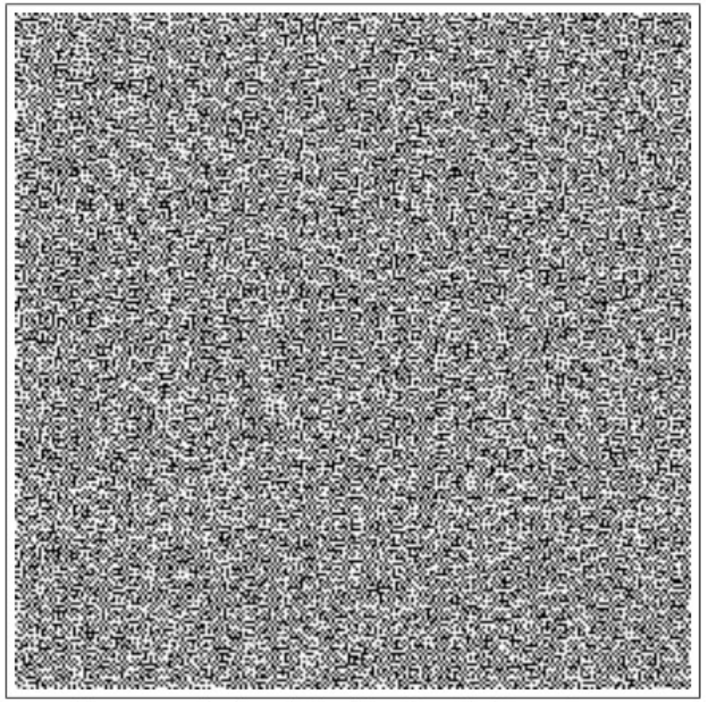


Ill-posed problems and regularization

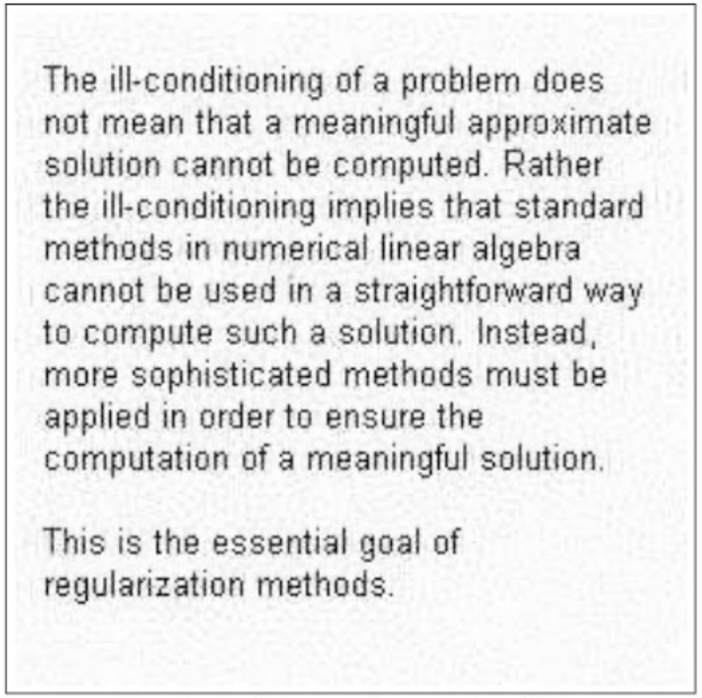
$$y = Kx + e$$
$$x^0 = K^{-1}y$$



Blurred image with noise



Inverse calculated by Fourier division, i.e., unregularized solution.



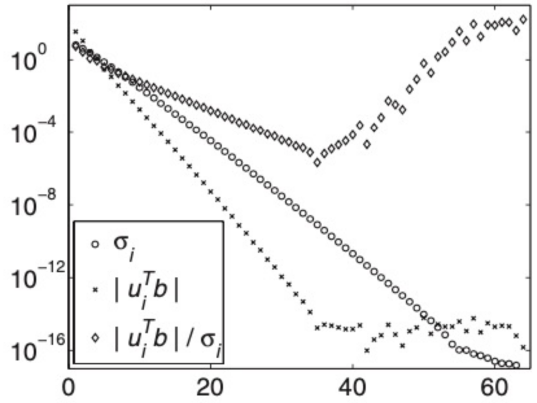
Regularized inverse image: $\lambda = 100$

Regularized solutions

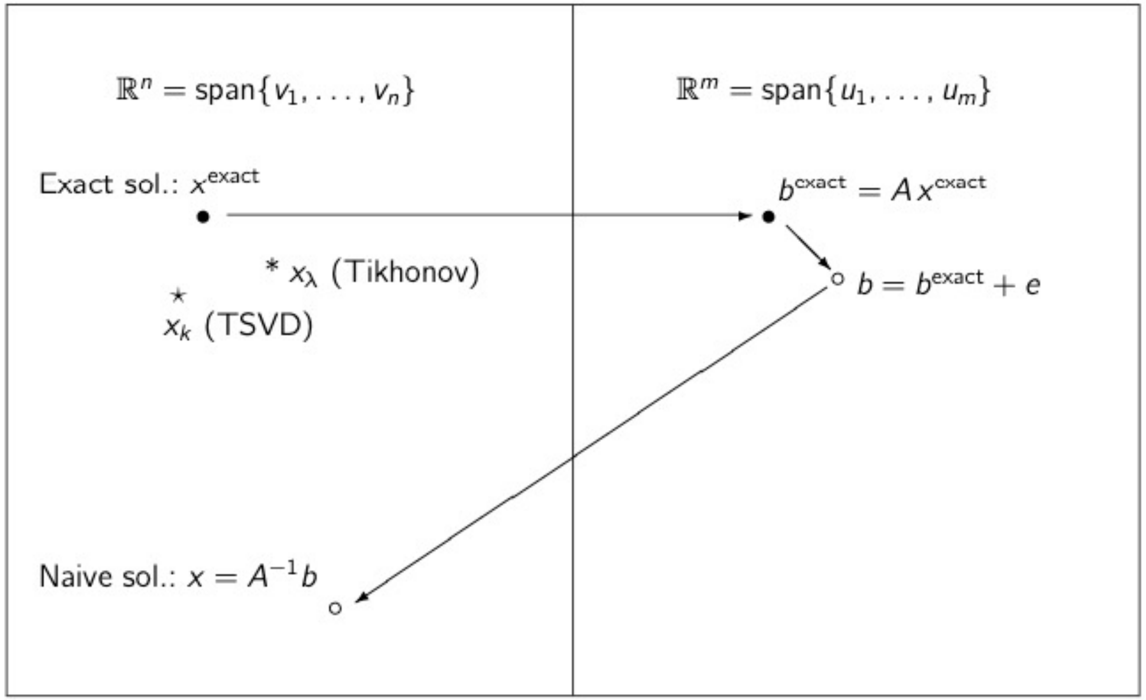
- Least squares
- Tikhonov regularization
- Truncated SVD
- Choice of reg. parameter
- Convergence analysis
- Optimality
- Data-driven methods

$$f(\lambda) = \arg\min_{f} \|y - Af\| + \lambda \mathcal{R}(f)$$

$$\tilde{f} = \arg\min \|y - Af\| \quad \text{subject to } \mathcal{R}(f) \leq E$$



e 3.6. *The Picard plot for the discretized gravity surveying*



From infinite to finite

- How to discretize an operator equation?
- Rudimentary convergence analysis

$$f(\lambda) = \arg\min_f \|y - Af\| + \lambda \mathcal{R}(f)$$

$$\tilde{f} = \arg\min \|y - Af\| \quad \text{subject to } \mathcal{R}(f) \leq E$$

Quadrature methods

$$\sum_{j=1}^{n} \omega_j K(s_i, t_j) \tilde{f}_j = g(s_i), \quad i = 1, \ldots, n$$

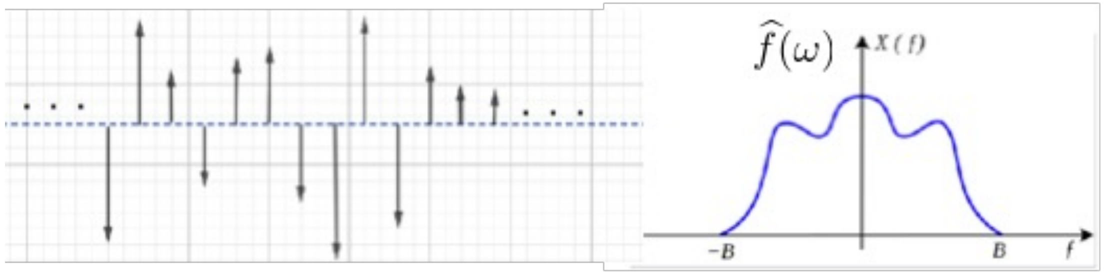
Galerkin / expansion methods

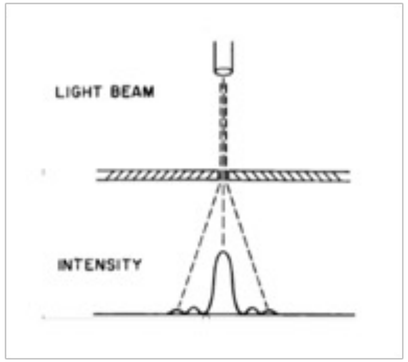
$$f^{(n)}(t) = \sum_{j=1}^{n} \zeta_j \phi_j(t)$$

$$\langle \psi_i, g \rangle = \sum_{j=1}^{n} \zeta_j \left\langle \psi_i, \int_0^1 K(s,t)\phi_j(t)dt \right\rangle, \quad i = 1, \ldots, n$$

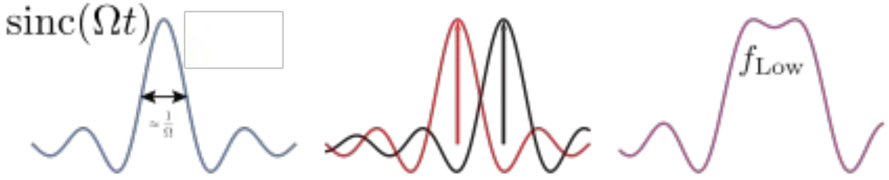
Super-resolution theory(ies)

Rayleigh-Nyquist limit





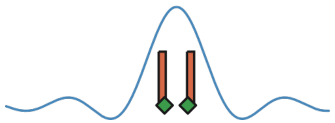
$$f(t) = \sum_{n=-\infty}^{\infty} f(nT)\,\mathrm{sinc}\left(\frac{t-nT}{T}\right) \text{ holds if } T < \frac{1}{2\Omega}$$



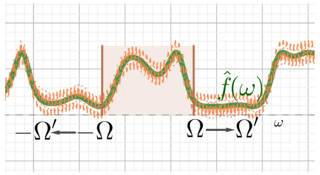
Classical resolution limit = inverse bandwidth

- Fast algorithms (FFT)
- Perfect stability

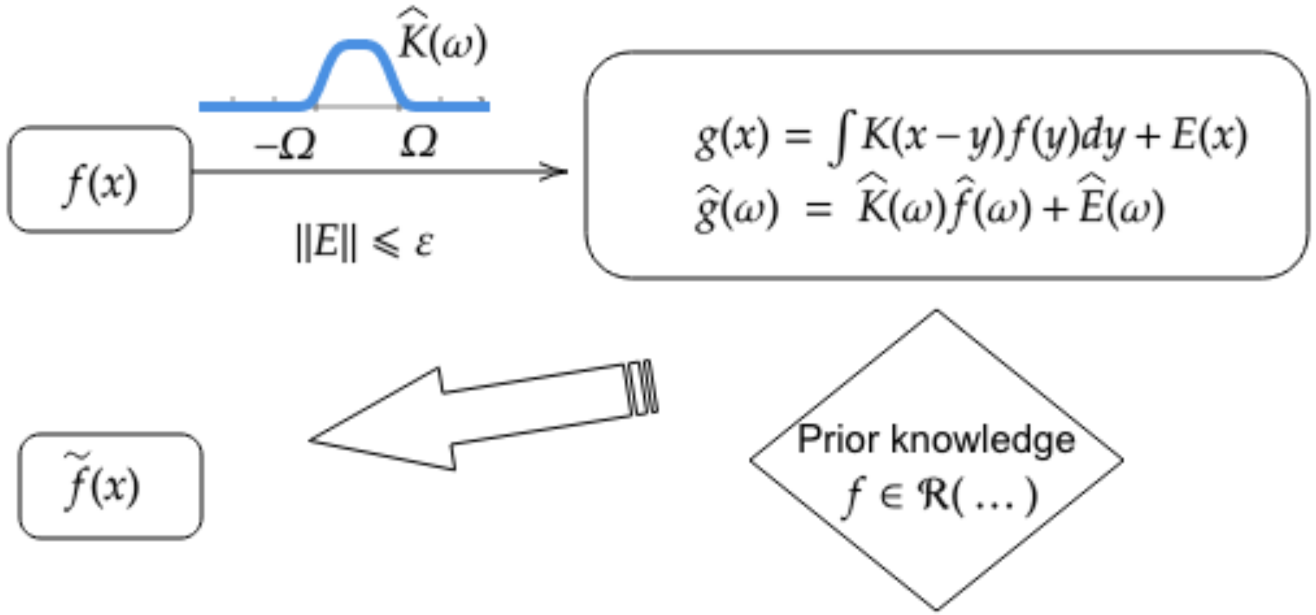
Inverse problems & Super-resolution



SR in time/space



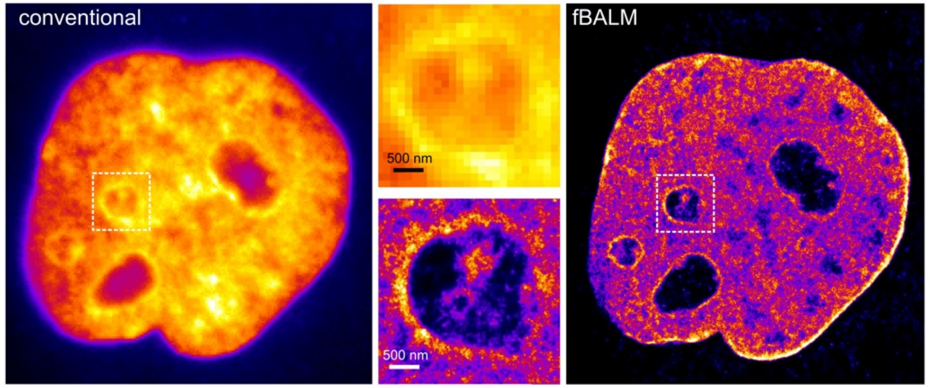
SR in frequency



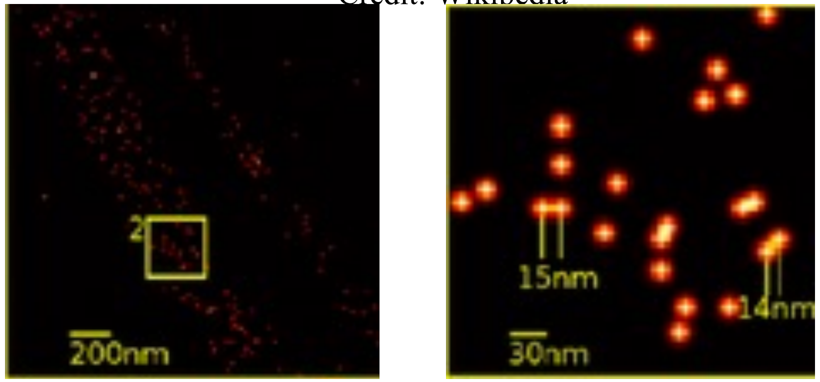
- Super-resolution (SR): recover $\hat{f}(\omega)$ for $|\omega| > \Omega$
- Eventual resolution limit depends on
  - Noise level $||E||$
  - Bandlimit Ω
  - Prior complexity/geometry…

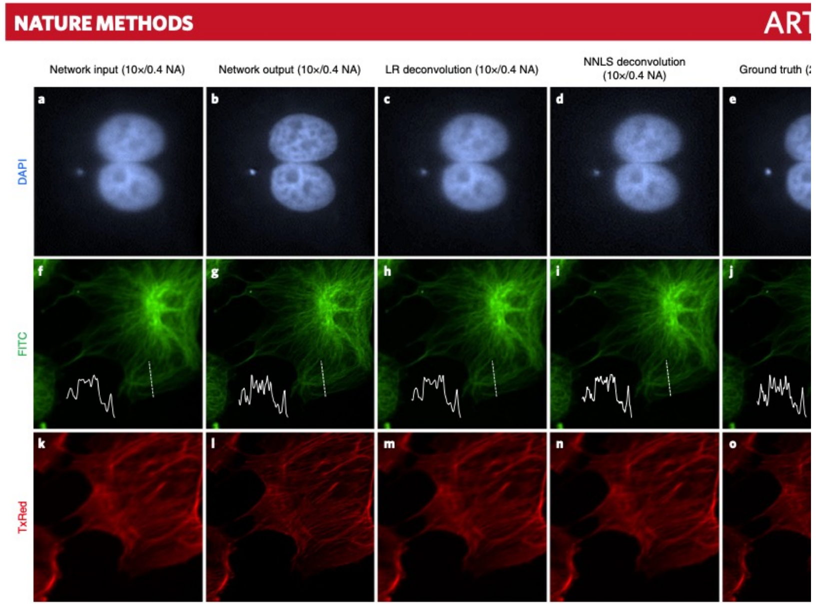
Q: how and when can we reliably extract high-res. information?

SR today

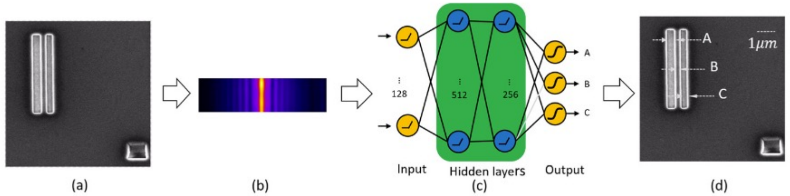


Credit: Wikipedia





Credit: *Nat. Methods* **16**, 103–110 (2019).

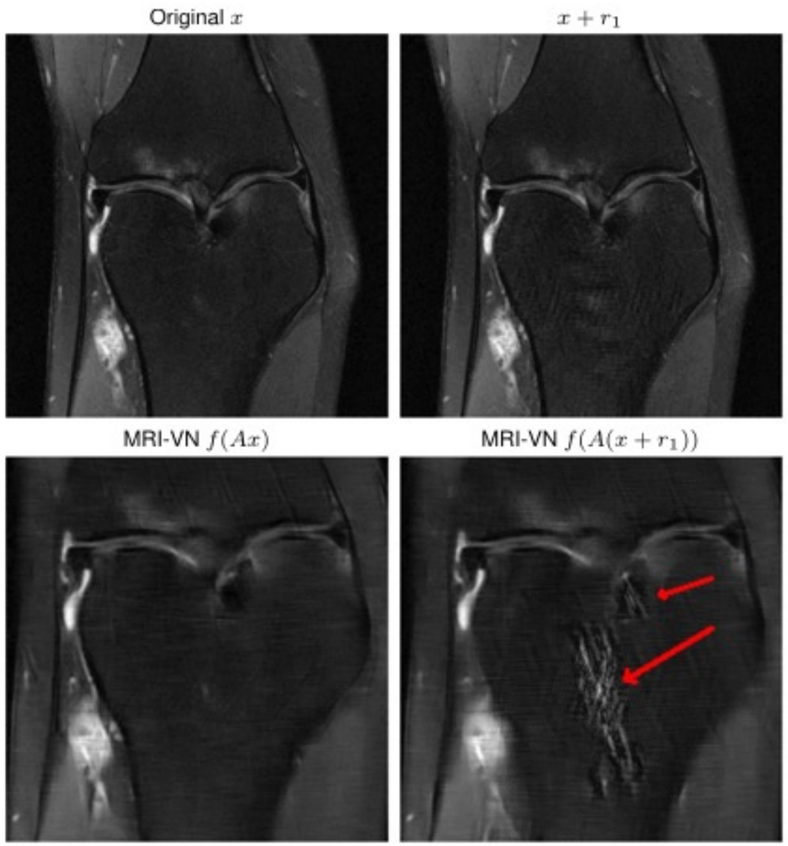


Credit: *Appl. Phys. Lett.116, 131105 (2020)*

HUGE SUCCESS! ...or is it??

Can we be sure?

* Artifacts in reconstruction
* Adversarial noise attacks



Credit: *Proc Natl Acad Sci USA* 201907377 (2020)

Rigorous SR guarantees largely lacking!

Classical/Linear SR

$$A\varphi = \sum_{i=1}^{\infty} \mu_i \langle \varphi, \varphi_i \rangle \psi_i$$

- "Classical" theory: depends on spectral properties of A
  - Singular value decay rates
  - Oscillation properties of singular functions
- Analytic continuation
- Super-resolution of compactly supported objects

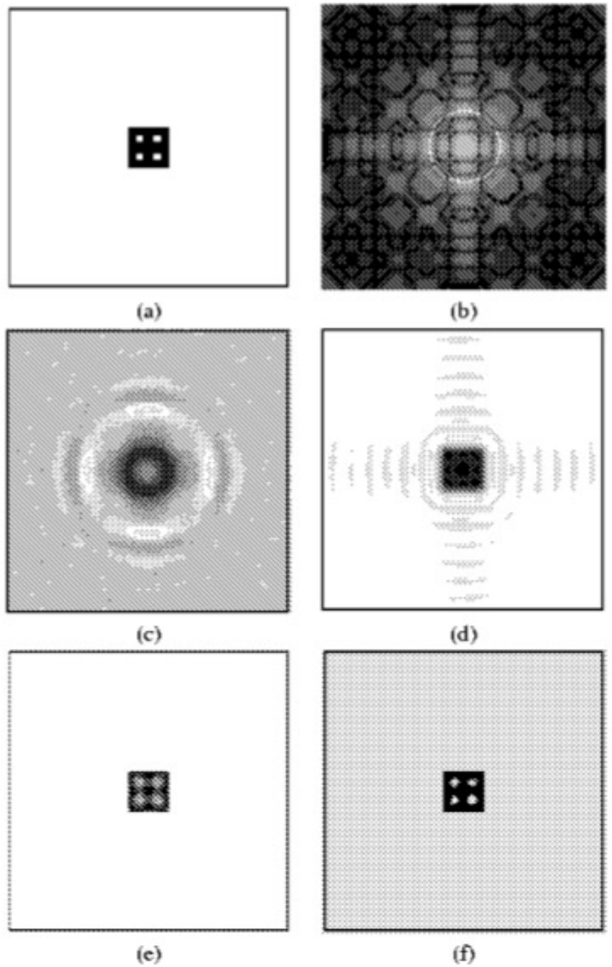
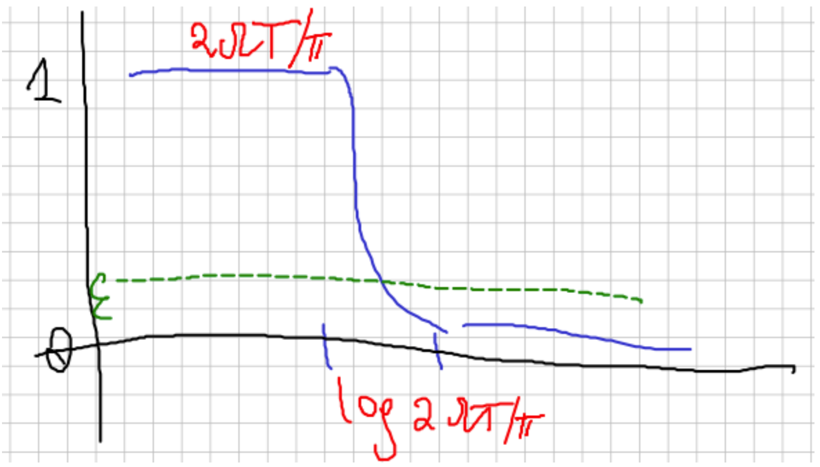


Figure 11.6. Example of super-resolution in far-field acoustic holography. (a) The object: a grid $2.5\lambda \times 2.5\lambda$ with bars 0.5λ wide. (b) The modulus of the FT of the object in (a); the white circle indicates the band of the far-field data. (c) The modulus of the noisy image at the distance 5λ from the object plane. (d) The reconstruction obtained by means of the inverse filtering method. (e) The reconstruction obtained by means of the iterative method with the constraint of bounded support. (f) The reconstruction obtained by means of the iterative method with the constraints of bounded support and positivity.

The linear setting for SR

- Unknown: $f \in L^2[-T,T]$ (prior information = compact support)
- Observe: Ω-bandlimited version of f
  - $A_{\Omega,T}: L^2[-T,T] \to L^2(\mathbb{R}) : A_{\Omega,T} f = \int_{-T}^{T} \frac{\sin(\Omega(t-s))}{\pi(t-s)} f(s)ds$
  - $c = \Omega T$, $(B_c f)(t) = \int_{-1}^{1} \frac{\sin(c(t-s))}{\pi(t-s)} f(s)ds, |t| < 1.$
- Eigenvalue distribution of B_c determines SR bounds

Landau&Widom 1980



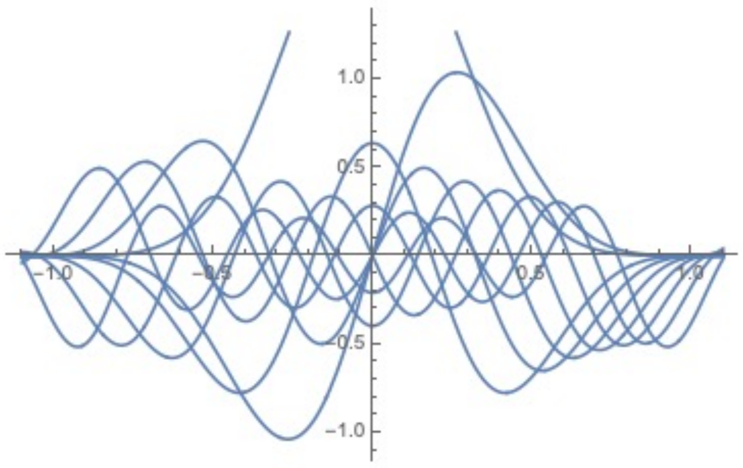
$$\boxed{\#\{\lambda: \lambda > \epsilon\} = \frac{2c}{\pi} + \frac{1}{\pi^2} \log \frac{1-\epsilon}{\epsilon} \log \frac{2c}{\pi} + o(\log c) \quad 0 < \alpha < 1, c \to \infty}$$

- ΩT is precisely the number of points in $[-T,T]$ taken at Nyquist rate
- As $T \to \infty$, the “super-resolution factor” vanishes ($\approx O(\log T/T)$)
- For fixed $T \gg 1$, we expect $\log \frac{1-\epsilon}{\epsilon} \approx \log \frac{1}{\epsilon}$ (as $\epsilon \to 0$) extra degrees of freedom
- In practice, may obtain significant SR for small c (i.e. T)

Prolate Spheroidal Wave Functions

In[28]:= `Plot[Table[SpheroidalPS[k, 0, 15, t], {k, 0, 8}], {t, -1.1, 1.1}]`

Out[28]=

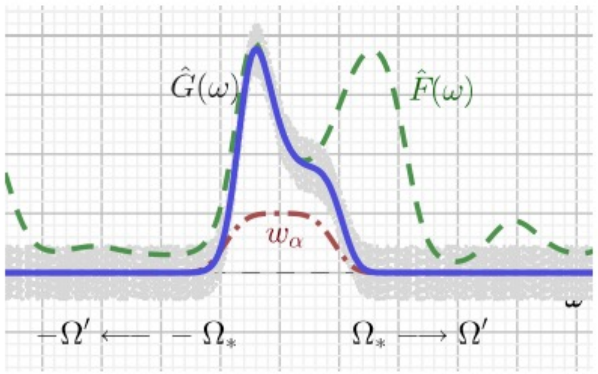


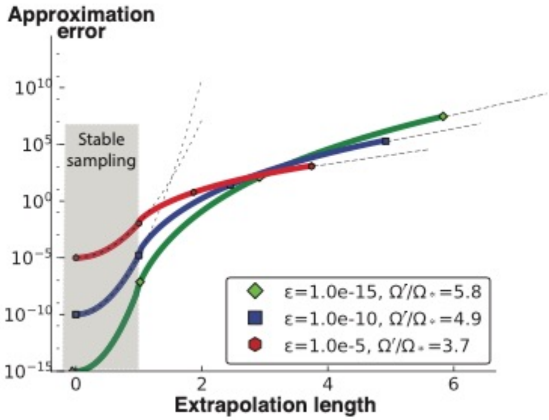
Sturm-Liouville theory

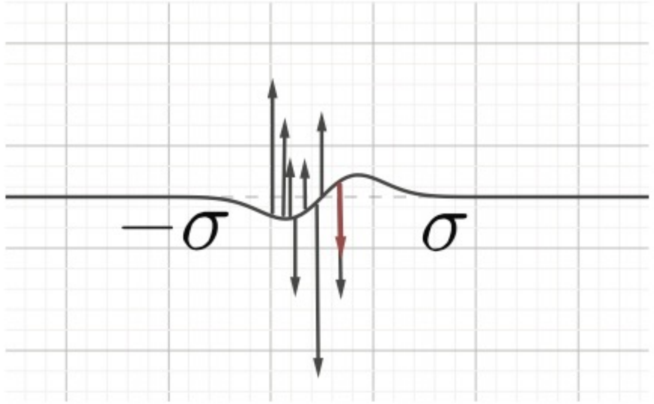
- $\frac{d}{dx}(1-x^2)\frac{d\psi(c;x)}{dx} + (\chi - c^2x^2)\psi(c;x) = 0$
- $\int_{-1}^{1} \frac{\sin(c(t-s))}{\pi(t-s)} \psi_n(c;s)ds = \lambda_n \psi_n(c;t)$
- Can be extended to $t \in \mathbb{R}$ by the above formula
- Simultaneously orthogonal on $\mathbb{R}$ and on $[-1,1]$
- $\widehat{\psi_n}$ is a rescaled copy of ψ_n

- Complete in $L^2[-1,1]$ and in PW_c (space of c-bandlimited functions)
- ψ_n has exactly n zeros in $[-1,1]$
- Can be computed efficiently
- Useful for extrapolation

"Soft" SR







$$G(y) = \int K(y-x)F(x)dx + E(x), \qquad ||E|| \le \varepsilon \qquad \operatorname{supp} F \subset [-\sigma, \sigma]$$

- Gaussian-type kernels: $\widehat{K}(\omega) = \exp(-|\omega|^\alpha)$
- Pointwise extrapolation estimate: $|\widehat{F}(\omega) - \widehat{F}_{rec}(\omega)| \asymp \varepsilon^{\gamma(\omega)}$
- Bandwidth extension $\Omega'/\Omega^* \asymp \frac{1}{\sigma} |\log \varepsilon|^{1-\frac{1}{\alpha}}$ (good for small objects)
- Bounds are minimax
- Key tools
  - Weighted approximation of bandlimited functions
  - Growth estimates of weighted polynomials

[**DB**, Demanet, Mhaskar, *Inverse Problems* 35, 015011 (2019).]

Sparsity constraints

From smoothness to sparsity

$$Ax = y, \quad A \in \mathbb{R}^{m \times n}, \quad m < n$$

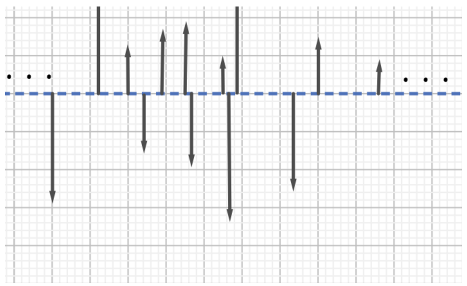
- Solution not unique in general
- Uniqueness restored if the number of nonzero elements of x is known to be small:

$$\|x\|_0 \leq k \ll n$$

- Compressed sensing: A is random

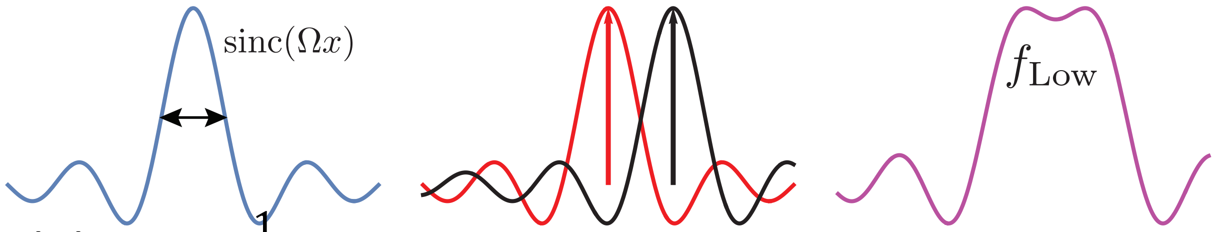
Model problem: spike deconvolution

High-resolution signal: $f_H = \sum_j \mu_j \delta(t - t_j)$

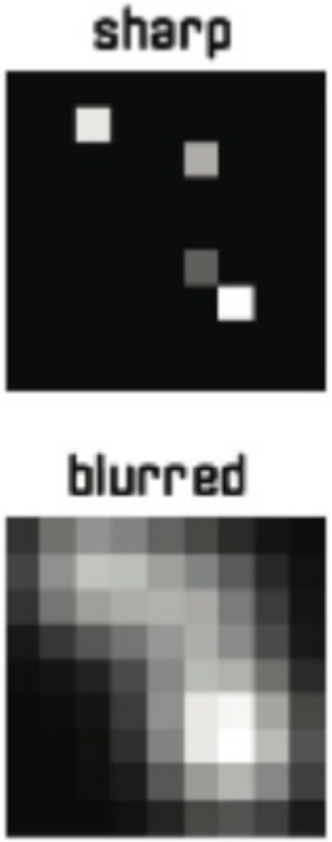


Bandlimited spectral measurements: $\hat{f}(\omega) = \sum_j \mu_j \exp(2\pi i \omega t_j), |\omega| \le \Omega$

Low-resolution signal:$F_{\mathrm{Low}} \approx \sum_j \mu_j \,\mathrm{sinc}(\Omega(t - t_j))$

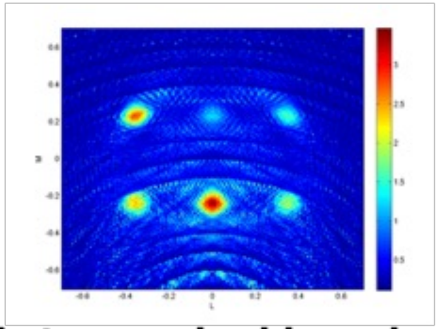


Rayleigh limit: $\sim \frac{1}{\Omega}$



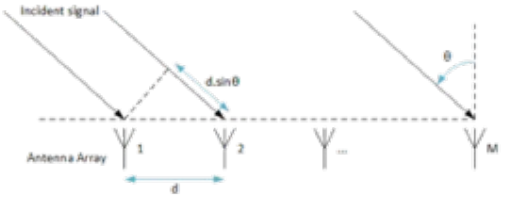
(One of) simplest nonlinear inverse problems

The sparsity prior



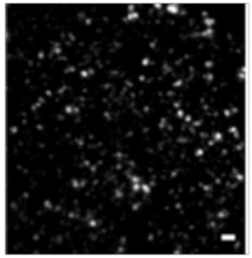
Astronomical imaging

Exponential fitting

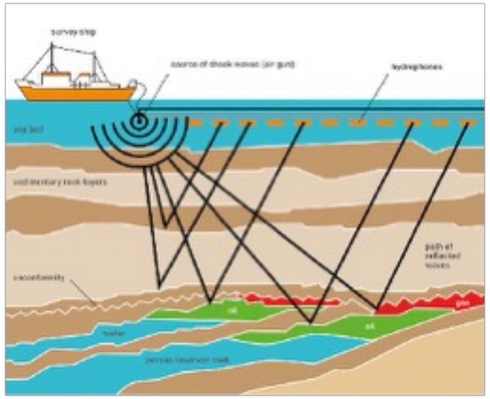


Direction of Arrival Estimation

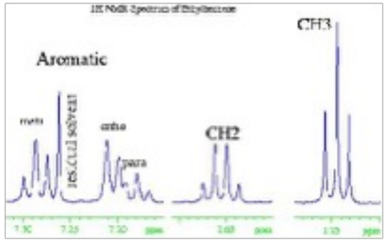
$$f(x) = \sum_{k=1}^{R} \alpha_k \delta(x - x_k)$$
$$\hat{g}(\omega) = \sum_{k=1}^{R} \alpha_k e^{\imath \omega x_k} + e(\omega), \quad |\omega| \leqslant \Omega$$



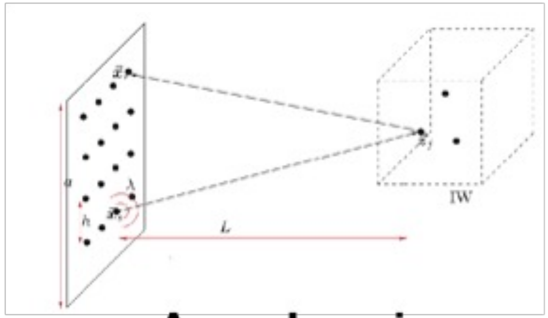
Fluorescence microscopy



Seismic inversion



NMR spectroscopy



Array Imaging

EXPERIMENTAL AND ANALYTICAL ESSAY
ON
THE EXPANSION PROPERTIES OF ELASTIC FLUIDS
AND ON THE FORCE OF EXPANSION OF
WATER VAPOR AND ALCOHOL VAPOR AT DIFFERENT
TEMPERATURES

By R. Prony, 1795
Translated by Dr. Ann Sanders/ETI

Gaspard-Clair-Francois-Marie
Riche de Prony
(1755-1839)

Prony's problem: recover $\{\mu_j, \rho_j\}$ from samples of

$$\tau(x) = \mu_1 \rho_1^x + \mu_2 \rho_2^x + \cdots \mu_n \rho_n^x$$

Gaspard Riche de Prony, *Essai experimental et analytique: sur les lois de la dilatabilite de fluides elastique et sur celles de la force expansive de la vapeur de l'alkool, a differentes temperatures*. **J. Ecole Polyt. 1:24-76 (1795).**

Prony's Insight

$$\tau(k) = \sum_{j=1}^{n} \mu_j \rho_j^k , \qquad \mu_j \neq 0, \rho_j \in \mathbb{C}$$

$$Q(x) = (x - \rho_1) \cdots (x - \rho_n) = x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

$$\boxed{\Rightarrow \sum_{\ell=0}^{n} c_\ell \tau(k + \ell) = 0, \qquad \forall k \in \mathbb{N}}$$

Proof: $\sum_{\ell=0}^{n} c_\ell \tau(k+\ell) = \sum_{\ell=0}^{n} c_\ell \sum_{j=1}^{n} \mu_j \rho_j^{k+\ell} = \sum_{j=1}^{n} \mu_j \rho_j^k \, Q(\rho_j) = 0$

Prony's Method

$$\tau(k) = \sum_{j=1}^{n} \mu_j \rho_j^k , \qquad \mu_j \neq 0, \rho_j \in \mathbb{C}$$

$2n$ unknowns

- Construct $n \times (n+1)$ **Hankel** matrix

$$H = [\tau(k+\ell)]_{k=0,1,\dots,n-1}^{\ell=0,1,\dots,n} = \begin{bmatrix} \tau(0) & \cdots & \tau(n) \\ \vdots & \ddots & \vdots \\ \tau(n-1) & \cdots & \tau(2n-1) \end{bmatrix}$$

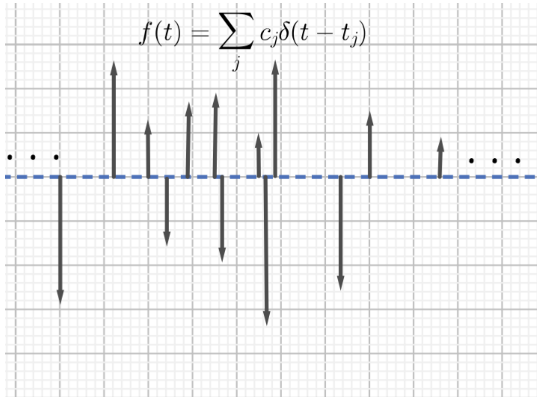
$2n$ samples

- Find $\boldsymbol{c} \in \ker(H)$, construct $Q_{\boldsymbol{c}}(x) = \sum_{j=0}^{n} c_j x^j$
- $\{\rho_j\}$ are the roots of $Q_{\boldsymbol{c}}$
- $\{\mu_j\}$ are given by

Vandermonde matrix

$$\begin{bmatrix} 1 & \cdots & 1 \\ \rho_1 & \cdots & \rho_n \\ \vdots & \ddots & \vdots \\ \rho_1^{n-1} & \cdots & \rho_n^{n-1} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} = \begin{bmatrix} \tau(0) \\ \tau(1) \\ \vdots \\ \tau(n-1) \end{bmatrix}$$

Parametric super-resolution



$$m(k) = \sum_{j=1}^{d} c_j \exp(2\pi i k t_j) + e(k),\, k = 0, \ldots, N$$

- 1D extensions
  - $m(k) \sim \sum_j f_j(k) \exp(ikt_j)$ for some parametric family $\{f_j\}$
  - $\mathfrak{D}(k)m(k) = 0$ for some <u>difference operator</u> $\mathfrak{D}(k)$ with poly. coefficients
  - $f = \sum_\lambda c_\lambda v_\lambda,\ Av_\lambda = \lambda v_\lambda,\ m(k) = F(A^k f)$
- N-D : $m(\boldsymbol{k}) \sim \sum_j c_j \exp(i\langle \boldsymbol{k}, \boldsymbol{t_j} \rangle)$
- "Algebraic data"

Existence/uniqueness?

Stability?

Golub et al. (1999)
Gustaffson et al. (2000)
Batenkov (2009)
Lasserre&Putinar (2015)
Kunis et al. (2015)
Comon&Usevich (2016)
Stampfer&Plonka (2019)
…

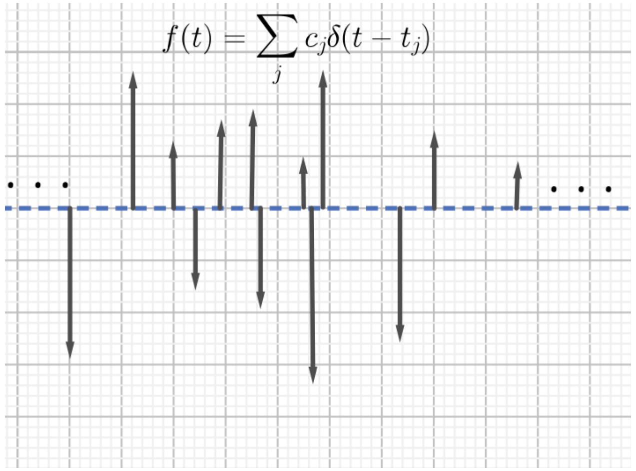
Stability analysis

Super Resolution Factor $= \dfrac{\textit{Coarse scale}}{\textit{Fine scale}}$

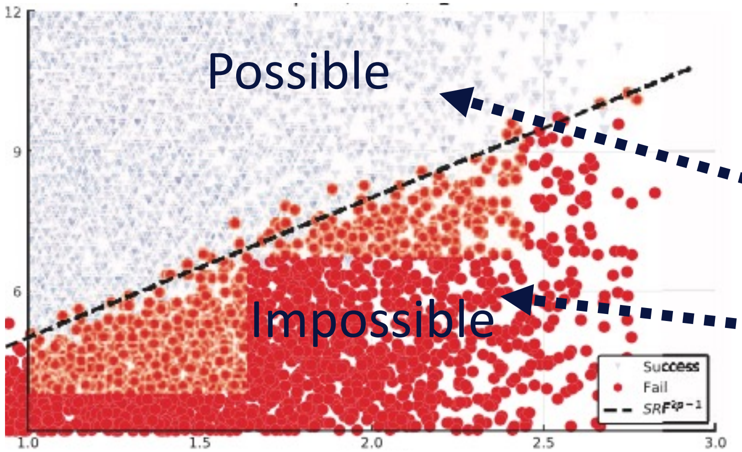
(SRF)

$$\Delta := \min_{j \neq k} |x_j - x_k|$$

$$\mathrm{SRF} := \frac{1}{\Delta\Omega} = \frac{\text{Rayleigh length}}{\Delta}$$

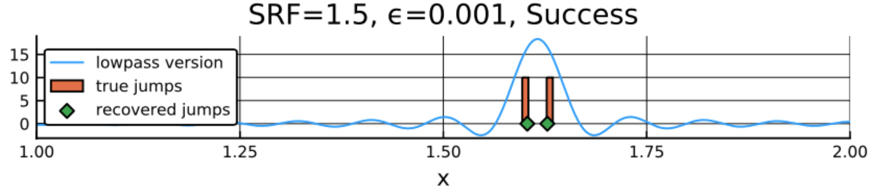


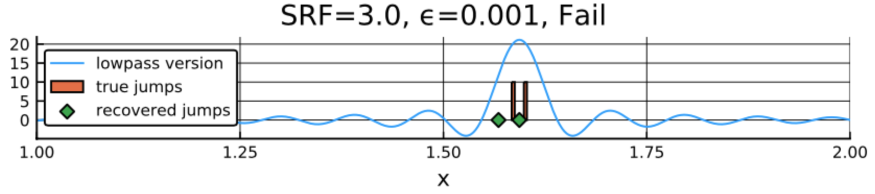
Noise



Optimal tradeoff

Error < separation/2

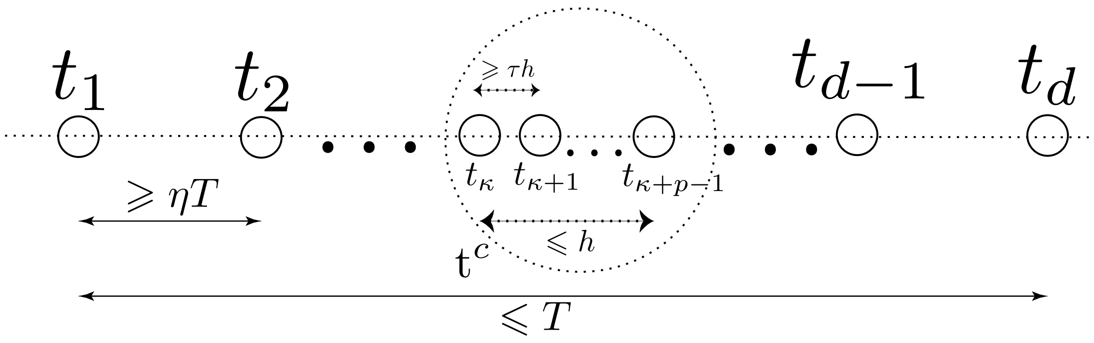




Error > separation/2

Batenkov et al., Information and Inference, **2020**

Main result

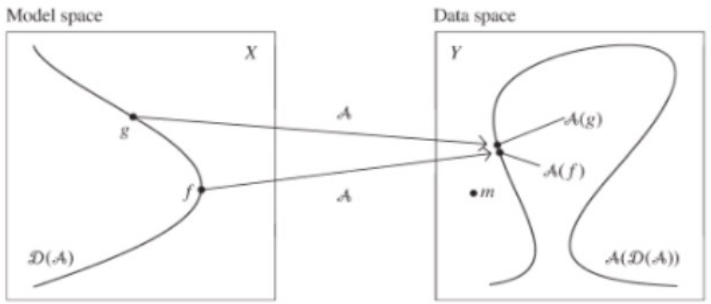


- Theorem: $\frac{c_1}{\eta T} \leq \Omega \leq \frac{c_2}{h}$, then with $\mathrm{SRF} := (\Omega h)^{-1}$ and $\epsilon < c_3 \mathrm{SRF}^{1-2p}$

$$|\tilde{t}_j - t_j| \asymp \frac{\epsilon}{\Omega} \times \begin{cases} \mathrm{SRF}^{2p-2}, & t_j \in \mathrm{t}^c \\ 1 & t_j \notin \mathrm{t}^c \end{cases}$$

$$|\tilde{c}_j - c_j| \asymp \epsilon \times \begin{cases} \mathrm{SRF}^{2p-1}, & t_j \in \mathrm{t}^c \\ 1 & t_j \notin \mathrm{t}^c \end{cases}$$

- Corollary: $\mathrm{SRF} \asymp \left(\frac{1}{\epsilon}\right)^{\frac{1}{2p-1}}$
- Main tools:
  - Local properties of the inverse mapping (spectrum of the Jacobian)
  - Quantitative Implicit function theorem

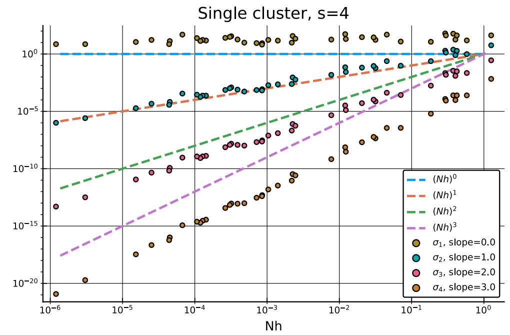


Vandermonde matrices

$$\mathbb{G}(\mathbf{x},\Omega) := \left[\mathrm{sinc}\left(\Omega\left(x_i - x_j\right)\right)\right]_{1\leqslant i,j\leqslant R}$$

$$\mathbf{V}_N(\mathbf{x}) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{2x_1} & e^{\imath x_2} & \cdots & e^{\imath x_R} \\ e^{\imath 2x_1} & e^{\imath 2x_2} & \cdots & e^{\imath 2x_R} \\ \vdots & \vdots & \vdots & \vdots \\ e^{\imath N x_1} & e^{\imath N x_2} & \cdots & e^{\imath N x_R} \end{bmatrix}$$

- Spectral properties of $\mathbb{G}, \mathbf{V}_N$ - crucial for SR



Connections to harmonic analysis

$$\|p\|_{L^q(I)} \leq \left\{\frac{A|I|}{\mu(E)}\right\}^{K-1} \|p\|_{L^q(E)}.$$

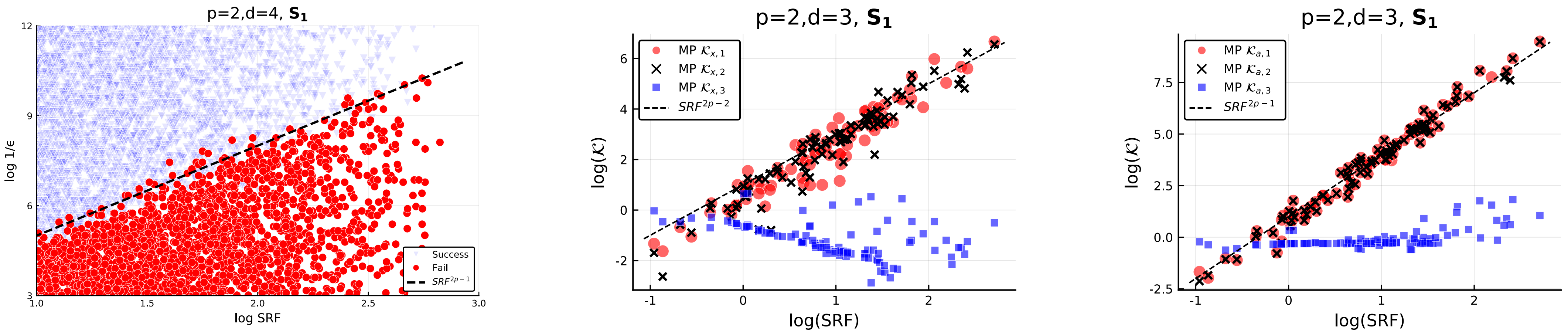
(Turan's Lemma)

$$p(t) = \textstyle\sum_{j=1}^{K} c_j \exp(\imath\lambda_j t)$$

$$E \subset I$$

Batenkov, D., Demanet, L., Goldman, G., & Yomdin, Y. *SIAM Journal on Matrix Analysis and Applications,* **2020**
Batenkov, D., Diederichs, B., Goldman, G., & Yomdin, Y. *Linear Algebra and its Applications,* **2021**
Batenkov, D., Goldman, G., *Applied and Computational Harmonic Analysis,* **2021**

The search for optimal computational methods



- “Algebraic” (Prony-based)
- Nonlinear least squares
- Homotopy (solving polynomials systems)
- Total variation relaxation

$$\arg\min_{\mu=(\boldsymbol{\alpha},\mathbf{x})} \parallel \mu \parallel_{TV} \ \ s.t. \parallel \hat{\mu} - \hat{g} \parallel \leq \varepsilon$$

- “Subspace” methods from signal processing

Problem not solved!

Candès, E. J., & Fernandez-Granda, C. (2014)
Cuyt, A., & Lee, W. (2018).
Morgenshtern, V. I. (2020).
Li, W., Liao, W., & Fannjiang, A. (2020).
Liu, P., & Zhang, H. (2020).

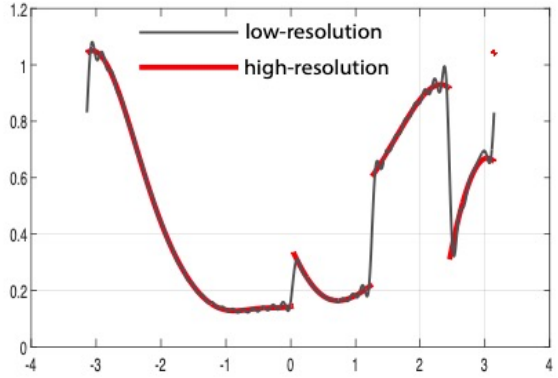
Towards hybrid modeling

Piecewise-smooth functions

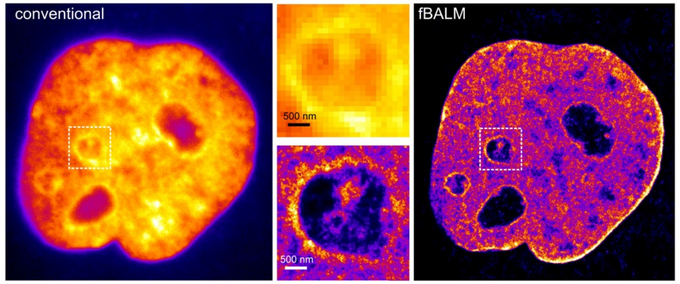
- If f is piecewise C^d then

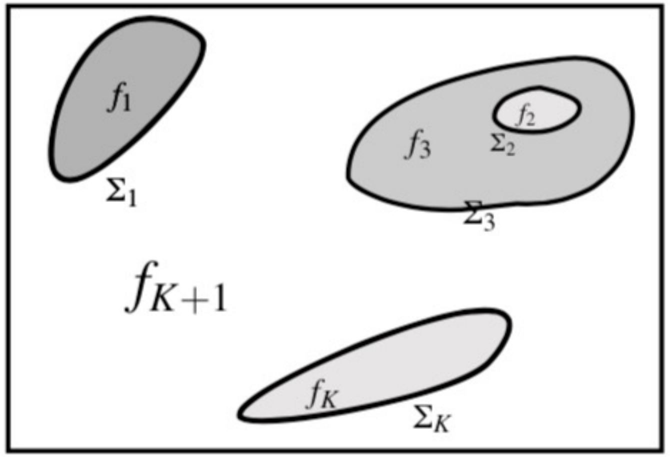
$$\hat{f}(N) \sim \sum_j \left(c_{j,0} + \frac{c_{j,1}}{N} + \cdots + \frac{c_{k,d}}{N^d}\right) e^{\imath t_j N} + O\left(N^{-d-1}\right)$$

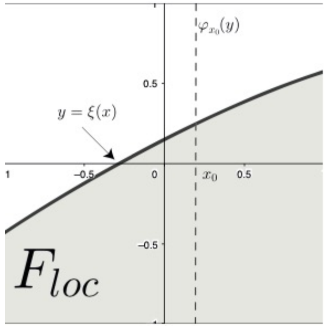
  - “Sparse+smooth” structure → cancellations
- Currently working on extension to 2D
- Next steps: applications for MRI imaging



[**DB**, Yomdin, Math.Comp.(2012)]
[**DB**, Math.Comp. (2015)]





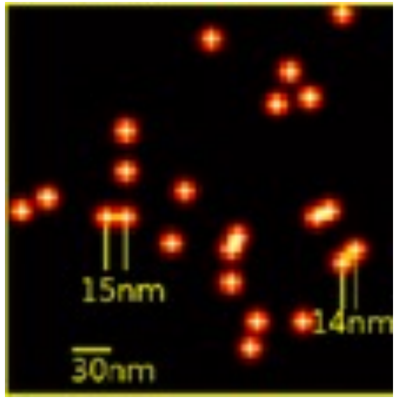


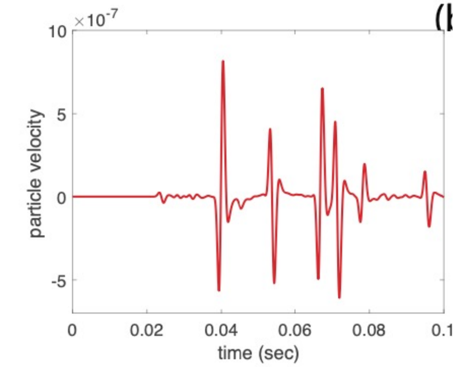
“Quasi-spikes”

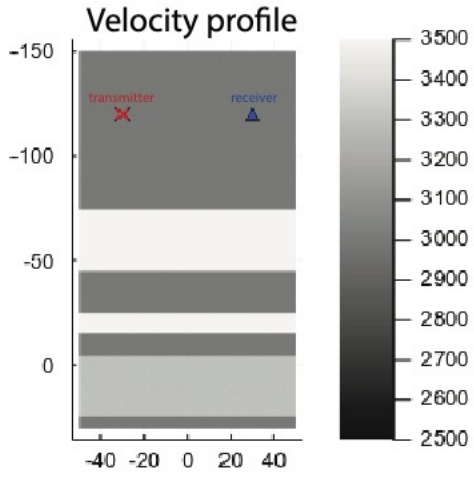
$$f \sim \sum_{k=1}^{R} f_k(x - x_k), supp\, f_k \subset [-\sigma, \sigma]$$

$$\Longrightarrow d(\omega) \sim \sum_{k=1}^{R} \hat{f}_k(\omega) e^{\imath \omega x_k} + e(\omega), \hat{f}_k \in B_\sigma$$

“Sparse+smooth” structure → cancellations

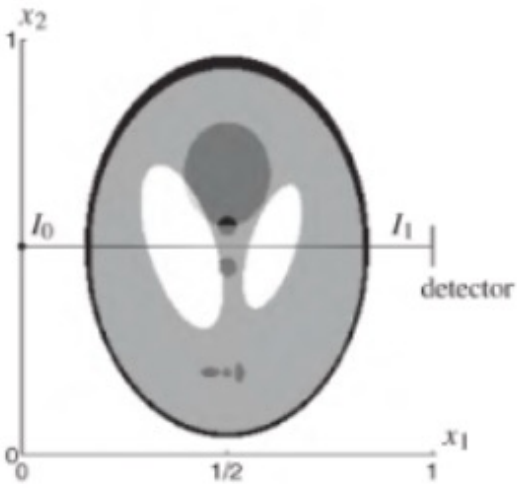


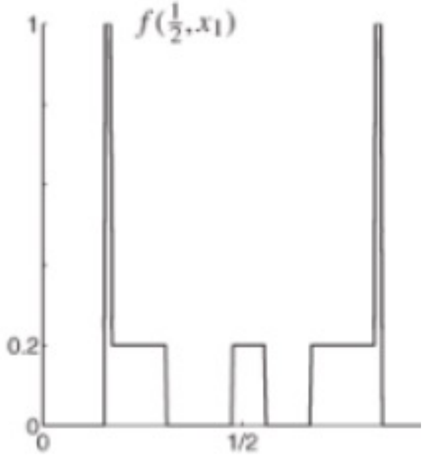


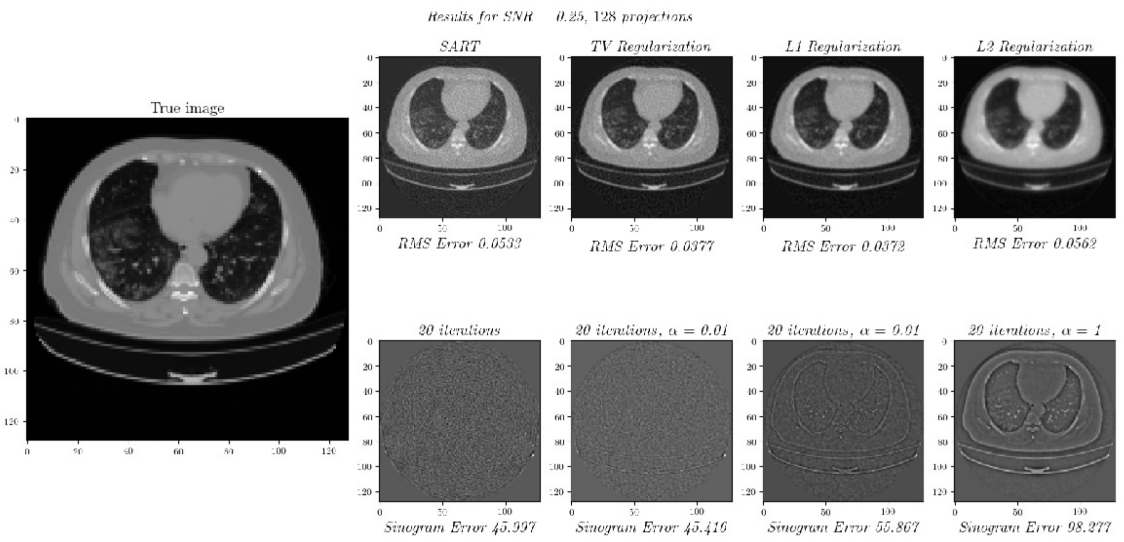


Applications

Transmission tomography (X-ray)





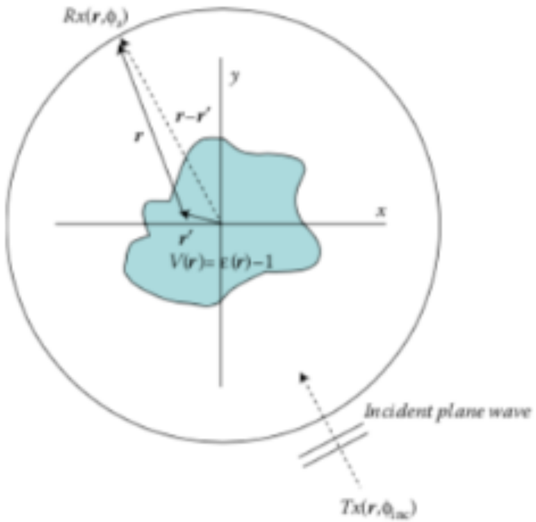


$$(Pf)(\omega, x) = \int_{-\infty}^{\infty} f(x + t\omega)dt, \; x \in \omega^{\perp}$$

- Limited-angle data problem (not all ω's are covered)

  - Radon transform
  - Filtered backprojection

Inverse scattering



$$\left(\Delta + \omega^2 m(x)\right) u_{\omega,f} = f(x), \qquad + \text{ boundary conditions}$$
$$d_{r,\omega,f} = u_{\omega,f}(x_r)$$

- find $m(x)$ from $d_{r,\omega,f}$
- nonlinear in general
- no satisfactory solution in 3D

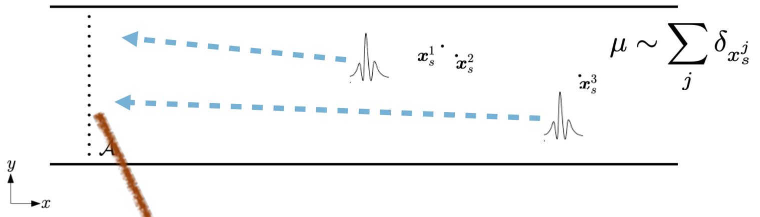
Data-driven models

- Can use a *neural network* to approximate the inverse mapping
- Excellent practical results given the correct architecture and sufficient training
- <u>Super-resolution/accuracy bounds in general unknown</u>
- Physics-informed Neural Networks

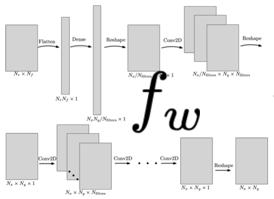
Imaging in waveguides



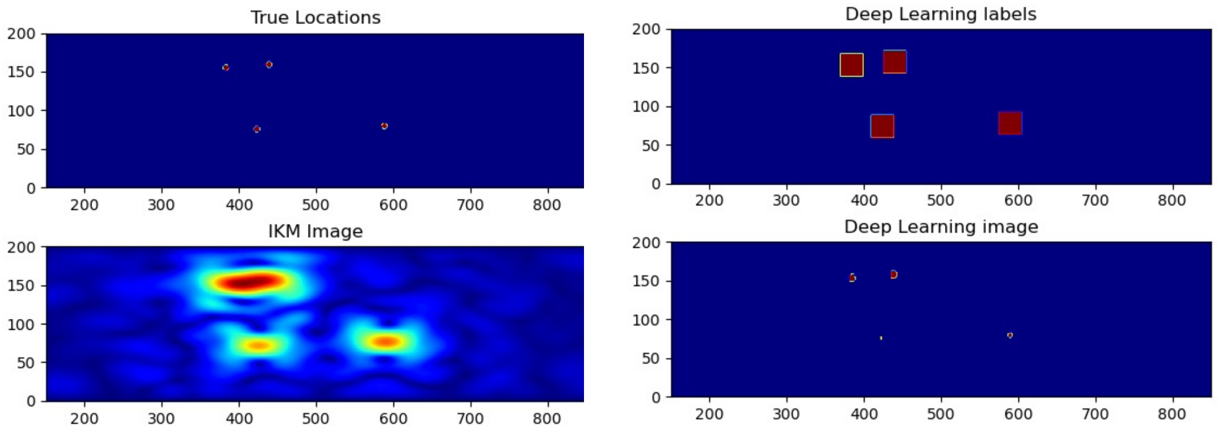

- Applications: fiber optics, underwater acoustics, seismics,…
- Our goal: overcome resolution limitations of traditional imaging methods
- Data-driven approach: Source Refocusing via Physics-Informed Neural-Network
- In progress: rigorous algebraic techniques



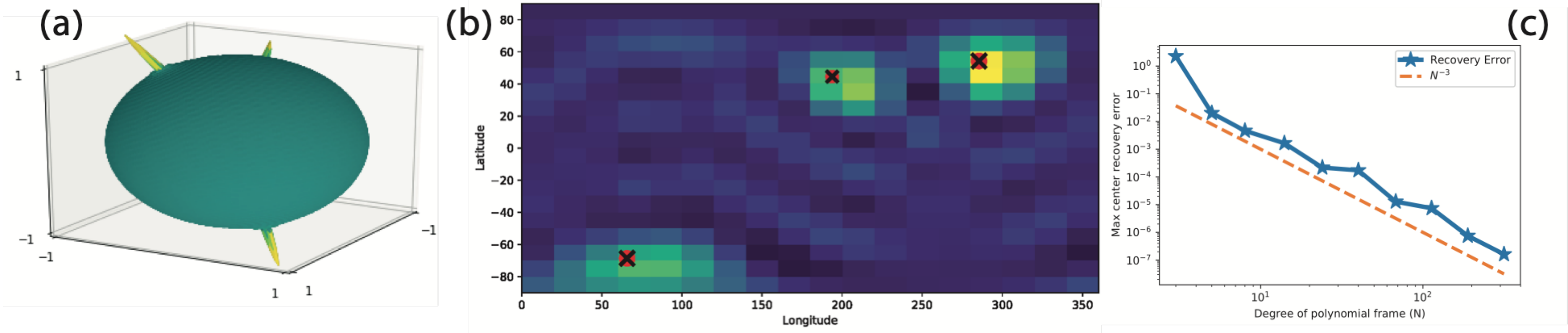
d_{low}



$$\min_w \|K_{low} * \mu - f_w(d_{low})\|^2 + \lambda\|K_{Hi} * \mu - \tilde{K} * f_w(d_{low})\|^2$$



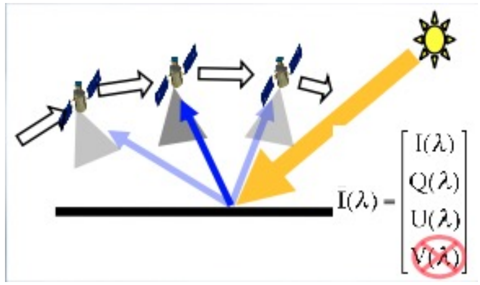
SR on manifolds



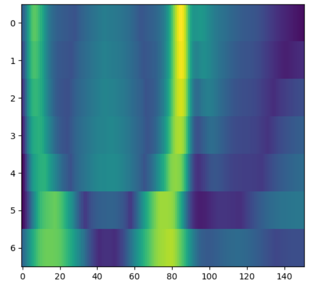
$$F \sim \sum_{k=1}^{K} \sum_{r=0}^{R} a_{k,r} G_r(x; x_k) \qquad G_r \text{ is the Green's function of } (I + \Delta)^{r/2}$$

- Setup: homogeneous Riemannian manifold
- Two-stage algorithm to recover F from low-frequency data
- High-accuracy localization of point sources
- Next steps: extensions to sub-manifolds

Remote sensing



λ



θ

- The inverse problem: retrieve cloud & aerosol optical properties

- Non-parametric approach: <u>**function approximation**</u> on unknown manifold

$$\hat{f}_k^{(n,q)}(x) := \frac{1}{M}\sum_{i=1}^{M} y_i \tilde{\Phi}_{n,q}\Big(\|x - x_i\|_{2,Q}\Big), \quad x \in \mathbb{R}^Q$$

  - <u>**Near state of the art results**</u>

- Work in progress: Physics-based Machine Learning approach

$$\hat{\mathbf{s}} \cdot \nabla I + (\mu_a + \mu_s) I = \mu_s \int p(\mathbf{s}', \hat{\mathbf{s}}) I(\boldsymbol{r}, \hat{\mathbf{s}}') \mathrm{d}\hat{\mathbf{s}}', \quad \boldsymbol{r} \in \Omega \quad + \text{ B.C.}$$

Administration

- Assumed knowledge: linear algebra, analysis, Fourier series
- References: will be provided for each topic
- Grade: 2-3 homework assignments + project
- Homework
  - Purpose: get hands-on experience with the topics
  - Code in any language
- Project
  - Read and summarize papers on a particular topic
  - Implement/analyze an algorithm
  - Combination of the above
  - More info during the semester