

# ALGEBRAIC RECONSTRUCTION OF PIECEWISE-SMOOTH FUNCTIONS OF TWO VARIABLES FROM FOURIER DATA

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**ABSTRACT.** We investigate the problem of reconstructing a 2D piecewise smooth function from its bandlimited Fourier measurements. This is a well known and well studied problem with many real world implications, in particular in medical imaging. While many techniques have been proposed over the years to solve the problem, very few consider the accurate reconstruction of the discontinuities themselves.

In this work we develop an algebraic reconstruction technique for two-dimensional functions consisting of two continuity pieces with a smooth discontinuity curve. By extending our earlier one-dimensional method, we show that both the discontinuity curve and the function itself can be reconstructed with high accuracy from a finite number of Fourier measurements. The accuracy is commensurate with the smoothness of the pieces and the discontinuity curve. We also provide a numerical implementation of the method and demonstrate its performance on synthetic data.

## 1. INTRODUCTION

In this work we revisit the classical problem of approximating a piecewise regular function  $F : \mathbb{T}^2 \rightarrow \mathbb{R}$  from Fourier data:

$$(1.1) \quad \widehat{F}(\omega_x, \omega_y) := \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F(x, y) e^{-i\omega_x x} e^{-i\omega_y y} dx dy \quad , \quad |\omega_x| \leq M, \quad |\omega_y| \leq N,$$

where  $\mathbb{T}^2$  is the 2D periodic torus  $[-\pi, \pi)^2$ . The presence of discontinuities hinders the otherwise fast convergence of the Fourier series of  $F$ , while also introducing the *Gibbs phenomenon* [37]. Both issues have very serious implications, e.g. when using spectral methods to calculate solutions of PDEs with shocks [41] or when approximating sharp edges, e.g. boundaries of tissues using Magnetic Resonance (MR) images obtained via the inverse Fourier transformation [66]. Therefore, an important questions arises:

*Can piecewise smooth functions be reconstructed from their Fourier measurements with accuracy comparable to their smooth counterparts?*

In the rest of the introduction, we first establish some notation (section 1.1), then present our main findings (section 1.2), and finally compare our results with prior works (section 1.3).

**1.1. Problem setting.** In our model we will develop a super-resolving technique (from Fourier measurements) for a piecewise-smooth 2D function  $F : \mathbb{T}^2 \rightarrow \mathbb{R}$  with only two continuity pieces (as described in fig. 1a) such that the boundary curve can be represented by the graph of a smooth function  $y = \xi(x)$  (as described in fig. 1b).

**Definition 1.** Let  $C_I^{(d+1)}$  denote the class of continuous functions having  $d$ -continuous derivatives in  $I \subseteq \mathbb{T}$ , such that  $f^{(d+1)}$  is piecewise-continuous and piecewise-differentiable in  $I$ .

**Definition 2.** Let  $PC_I^{(d+1, K)}$  denote the class of functions  $f$  with  $K$  discontinuity points,  $-\pi \leq \xi_1 < \xi_2 < \dots < \xi_K < \pi$  s.t.  $f \in C_{[\xi_i, \xi_{i+1}]}^{(d+1)}$  for  $1 \leq i \leq K - 1$ .

**Definition 3.** Let  $F : \mathbb{T}^2 \rightarrow \mathbb{R}$ . For each  $x \in \mathbb{T}$  let  $F_x : \mathbb{T} \rightarrow \mathbb{R}$  denote the “slice”  $F_x(y) := F(x, y)$ , and assume that for each  $x \in \mathbb{T}$  we have  $F_x \in PC_{\mathbb{T}}^{(d+1, 1)}$ . Also denote the smooth boundary curve,  $\Sigma$ , as a

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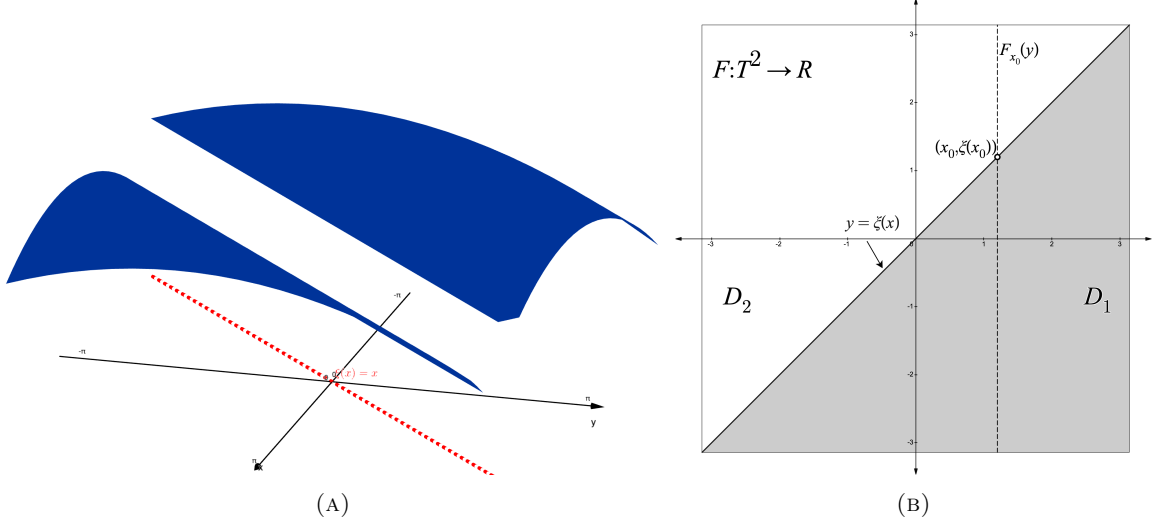


FIGURE 1. Our model of the piecewise-smooth 2D function  $F : \mathbb{T}^2 \rightarrow \mathbb{R}$  with two continuity pieces. (1a) The graph of  $F$ . (1b) The continuity domains  $D_1, D_2$  and the jump curve  $\Sigma$ .

function of  $x$  by:

$$(1.2) \quad \Sigma = \{(x, y) : y = \xi(x), x \in \mathbb{T}\}$$

and assume that  $\xi(x) \in C_{(-\pi, \pi)}^{d_\xi}$  for some  $d_\xi \in \mathbb{N}$ ,  $d_\xi \geq d$ .

Now we define the two continuity pieces of  $F$  and their domains:

$$(1.3) \quad \begin{aligned} F_1 : D_1 &\rightarrow \mathbb{R}, \text{ where } D_1 := \{(x, y) \mid -\pi \leq y \leq \xi(x)\} \\ F_2 : D_2 &\rightarrow \mathbb{R}, \text{ where } D_2 := \{(x, y) \mid \xi(x) < y < \pi\}. \end{aligned}$$

Our function  $F$  can therefore be written as follows:

$$(1.4) \quad F(x, y) = \begin{cases} F_1(x, y) & \text{if } (x, y) \in D_1 \\ F_2(x, y) & \text{if } (x, y) \in \mathbb{T}^2 \setminus D_1. \end{cases}$$

**Remark:** note that the assumptions above imply that each slice  $F_x$  has a single jump at  $y_x = \xi(x)$ , and is otherwise periodic and smooth. This assumption can be relaxed so that  $F_x$  has also a jump at the endpoints, but we will not consider this case here.

In this paper, we present a technique that, given any  $x \in \mathbb{T}$ , reconstructs the slice  $F_x(y)$  by approximating the jump location  $y_x = \xi(x)$  and the jump magnitudes of  $\frac{\partial^k}{\partial y^k} F_x(y)$  at  $y_x$  (see eq. (1.5) below) for  $0 \leq k \leq d_r$  where  $d_r \leq d$  is the chosen reconstruction order. Our proposed method utilizes and combines the two methods described in [15, 10] (which will be presented and analyzed going forward).

**1.2. Summary of our results.** To state our results, let us first establish some additional notation related to our model.

**Definition 4.** Let  $F : \mathbb{T}^2 \rightarrow \mathbb{R}$  with  $F_x$  as described in definition 3 where  $x \in \mathbb{T}$ , and let  $A_l(x)$  denote the jump magnitude of  $\frac{d^l}{dy^l} (F_x)$  at  $y_x := \xi(x)$ :

$$(1.5) \quad A_l(x) := \lim_{y \rightarrow y_x^+} \frac{\partial^l}{\partial y^l} F_x(y) - \lim_{y \rightarrow y_x^-} \frac{\partial^l}{\partial y^l} F_x(y).$$

We further denote:

- $M_\xi := \sup_{x \in \mathbb{T}} \left\{ \left| \frac{d^l}{dx^l} \xi(x) \right| \mid l = 0, \dots, d \right\} < \infty.$
- $B_F := \sup_{y \in \mathbb{T} \setminus \{\xi(x)\}} \left\{ \left| \frac{\partial^{k+1}}{\partial x^{k+1}} F_x(y) \right| \mid k = 0, \dots, d+1 \right\} < \infty, \text{ for all } x \in \mathbb{T}.$

- $A_x := \max_{l=0, \dots, d+1} \{|A_l(x)|\} < \infty$ .

We next define the  $\psi_{\omega_y}$  function, which is an essential ingredient in our work.

**Definition 5.** Let  $F : \mathbb{T}^2 \rightarrow \mathbb{R}$  be an unknown 2D piecewise-smooth function as defined in definition 3. Assume that the given data are the  $(2M + 1) \times (2N + 1)$  Fourier coefficients of  $F$ . For each  $|\omega_y| \leq N$  we define

$$(1.6) \quad \begin{aligned} \psi_{\omega_y}(x) &:= \widehat{F}_x(\omega_y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iy\omega_y} F(x, y) dy \\ &= \frac{1}{2\pi} \left( \int_{-\pi}^{\xi(x)} e^{-iy\omega_y} F_1(x, y) dy + \int_{\xi(x)}^{\pi} e^{-iy\omega_y} F_2(x, y) dy \right) \end{aligned}$$

where  $x \in \mathbb{T}$ .

**Lemma 1.1.** Let  $\xi$  as in eq. (1.2) s.t.  $\xi \in C_{(-\pi, \pi)}^{d_\xi}$  where  $d_\xi \geq d$ , then for each  $\omega_y \in \mathbb{Z}$ ,  $\psi_{\omega_y}$  satisfies the following properties:

- $\psi_{\omega_y} \in PC_{\mathbb{T}}^{(d+1, 1)}$ .
- $\psi_{\omega_y}$  and its first  $d$  derivatives have a (single) jump discontinuity at  $x = -\pi$ .

The proof of Lemma 1.1 is presented in the appendix. For further developments, we denote by  $A_l^\psi(\omega_y)$  the jump magnitude of  $\frac{d^l}{dx^l}(\psi_{\omega_y})$  at  $x = -\pi$ :

$$(1.7) \quad A_l^\psi(\omega_y) := \psi_{\omega_y}^{(l)}(-\pi) - \psi_{\omega_y}^{(l)}(\pi).$$

The crux of our method is the following. In [15, 10] it was shown that a piecewise-smooth 1D function  $f \in PC_I^{(d+1, K)}$  can be reconstructed from its Fourier coefficients with high accuracy by a certain “algebraic” reconstruction procedure (algorithm 2). Our idea is to use this 1D method for each  $x \in \mathbb{T}$  to approximate  $F_x \in PC_{\mathbb{T}}^{(d+1, 1)}$ , and for doing so, we need an accurate approximation of  $\widehat{F}_x(\omega_y) \equiv \psi_{\omega_y}(x)$ . For this latter task, we can again use the 1D method, provided sufficiently many Fourier coefficients of  $\psi_{\omega_y} \in PC_{\mathbb{T}}^{(d+1, 1)}$ . However, those are given by our data

$$(1.8) \quad \widehat{F}(\omega_x, \omega_y) \equiv \widehat{\psi_{\omega_y}}(\omega_x).$$

To see why (1.8) is true, consider the definition

$$\widehat{F}(\omega_x, \omega_y) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F(x, y) e^{-ix\omega_x} e^{-iy\omega_y} dy dx.$$

Using Fubini’s theorem [34] and (1.6) we have

$$\begin{aligned} \widehat{F}(\omega_x, \omega_y) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ix\omega_x} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iy\omega_y} F(x, y) dy dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi_{\omega_y}(x) e^{-ix\omega_x} dx = \widehat{\psi_{\omega_y}}(\omega_x). \end{aligned}$$

**Outline of method:** For a function  $F : \mathbb{T}^2 \rightarrow \mathbb{R}$  as in definition 3 we require its  $(2M + 1) \times (2N + 1)$  Fourier coefficients as the input data. Then for every  $|\omega_y| \leq N$  we approximate  $\psi_{\omega_y}$  (see eq. (1.6)) using  $\{\widehat{F}(\omega_x, \omega_y)\}_{|\omega_x| \leq M}$  as the input for our adaptation of the algorithms described in [15, 10] which will yield an approximation  $\widehat{\psi_{\omega_y}}$  for  $F_x$ ’s Fourier coefficients at  $x \in \mathbb{T}$ . Next we take this approximation of  $\{\widehat{F}_x(\omega_y)\}_{|\omega_y| \leq N}$  and use it as an input for our second part of the two-part algorithm to obtain an approximation  $\widehat{F}_x$  to the slice  $F_x$ . The entire algorithm is presented in algorithm 3 below.

Our main theoretical result provides upper bounds on the accuracy of recovering each  $F_x$ , including the position of the discontinuity curve  $\Sigma$  and the magnitudes  $A_l(x)$  of the jump discontinuities of  $\frac{\partial^l}{\partial y^l} F_x$  at  $y = \xi(x)$ , as well as the pointwise accuracy of approximating  $F_x$  for each  $x \in \mathbb{T}$ , using the method outlined above.

**Theorem 1.** Let  $F : \mathbb{T}^2 \rightarrow \mathbb{R}$  as in definition 3 with  $(2M + 1) \times (2N + 1)$  Fourier coefficients as in eq. (1.1) such that  $\forall x \in \mathbb{T}$  we have  $F_x \in PC_{\mathbb{T}}^{(d+1,1)}$  as in definition 4, and assume the following relation between  $M$  and  $N$ :

$$(1.9) \quad N^2 \leq M.$$

Further assume that  $\inf_{x \in \mathbb{T}} |A_0(x)| = A_L > 0$ . There exist constants  $\mathcal{R}_{\Gamma|\gamma}$ ,  $C_{2,d}$ ,  $C_{6,d-l}$ ,  $C_{10,d}$  (as defined in eqs. (2.58), (2.61) and (2.67)) such that for every  $r > 0$  there exists  $N'(r)$  such that for all  $N > N'$  and  $M \geq N'^2$  the following bounds hold for all  $x \in \mathbb{T}$ :

$$(1.10) \quad \begin{aligned} \left| \widetilde{\xi}(x) - \xi(x) \right| &\leq C_{2,d} \frac{A_x}{|A_0(x)|} \cdot \mathcal{R}_{\Gamma|\gamma} \cdot N^{-d-2} \\ \left| \widetilde{A}_l(x) - A_l(x) \right| &\leq C_{6,d-l} \cdot \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} \cdot N^{l-d-1} \\ \left| \widetilde{F}_x(y) - F_x(y) \right| &\leq C_{10,d} \frac{A_x(1+A_x)}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{-d-1}, \quad \forall y \in \mathbb{T} \setminus B_r(\xi(x)), \end{aligned}$$

where  $A_x$  and  $A_0(x)$  are as in definition 4, and  $B_r(t)$  is a ball of radius  $r$  centered at  $t$ .

The key estimate used to prove theorem 1 is the following reconstruction bound on each  $\psi_{\omega_y}$  (recall definition 5):

**Theorem 2.** Let there be given  $(2M + 1) \times (2N + 1)$  Fourier coefficients of  $F$  as in eq. (1.1). Let  $|\omega_y| \leq N$  be fixed. Then there exist  $\mathcal{H}$ ,  $\mathcal{H}_l$ ,  $\mathcal{H}_T^*$  and  $R_d$  (as defined respectively in eqs. (2.24), (2.25), (2.31) and (2.35)) s.t.:

$$(1.11) \quad \begin{aligned} \left| \widetilde{A}_l^\psi(\omega_y) - A_l^\psi(\omega_y) \right| &\leq \mathcal{H}_{d-l} A_x |\omega_y|^d M^{l-d-1}, \quad l = 0, \dots, d, \\ \left| \widetilde{\psi}_{\omega_y}(x) - \psi_{\omega_y}(x) \right| &\leq (\mathcal{H}_T^* + R_d \mathcal{H}) A_x |\omega_y|^d M^{-d-1}. \end{aligned}$$

As one can see, the decay rate in the bound for the error in approximating the jump magnitudes or in the pointwise accuracy for approximating  $\psi_{\omega_y}(x)$  is effected by the number of Fourier coefficients that is used as input and the order of reconstruction. Also we notice the effect of a higher order reconstruction,  $l$ , diminishes the accuracy of the jump magnitudes approximation, but all of that is to be expected and seen in other work on this subject (see [15, 10]). A new phenomenon appears in the 2D case: as  $N$  increases, the error bounds in(1.11) increase as  $N^d$  for a fixed  $M$ . Therefore we impose the additional assumption in (1.9) which ensures sufficiently fast decay in (1.10). This is most certainly an artifact of our method as we are reconstructing the slices in a fixed  $x$  direction.

**1.3. Related work.** In this section we provide some background and review related literature on the problem of high-accuracy reconstruction of piecewise-smooth functions.

It is well-known that linear trigonometric approximation of smooth periodic functions is minimax optimal on the torus  $\mathbb{T}^q$  in the sense of  $n$ -widths [54]. In particular, for functions with  $r$  continuous derivatives (more precisely, for which  $\Delta^{r/2} f$  is bounded in  $\ell^p$  where  $\Delta$  is the Laplacian), the approximation error by trigonometric polynomials of total degree  $n$  (a space of dimension  $O(n^q)$ ) decays as  $n^{-r}$ , with the optimal rate achieved by a particular class of summability kernels. This result is valid for  $p = \infty$  as well. For functions with singularities, the approximation error decays very fast “away from the discontinuities”; however, the discontinuities themselves can be localized with resolution at most  $1/n$ .

Reconstruction of piecewise-smooth functions in one variable from Fourier data has been investigated extensively. The reader is referred to [15, 10] for a detailed discussion of the problem, its history, and a solution based on parametric reconstruction of the piecewise-polynomial approximation through algebraic techniques. This method, in turn, is based on an earlier work by K.Eckhoff [32], introducing an essential modification: the algebraic system is solved for a small subset of the entire Fourier data, to attain maximal accuracy. The present work extends these ideas into the two-dimensional case (see also related work [7] on reconstruction from moments). The one-dimensional algebraic reconstruction technique in [15, 10] is closely related to exponential fitting and the inverse problem of recovering sparse measures and distributions from low-frequency data, cf. [3, 22, 8, 9, 12, 16, 11, 13, 17, 18, 14, 43, 44, 45, 33, 55] and some multi-dimensional generalizations thereof [46, 53, 59, 27, 25, 28, 60, 61]; providing robustness estimates for such problems is a nontrivial task. Fourier reconstruction of non-periodic smooth functions (i.e. with endpoint singularities, as our  $\psi_{\omega_y}$ , recall Lemma 1.1) is a classical topic in numerical analysis of spectral methods [40, 38, 39, 23], see also [48, 49, 2, 6, 58].

In [36] the spectral edge detection method by Gelb&Tadmor [35, 65] utilizing annihilating filters is extended to two dimensions and nonuniform Fourier samples (see also [1]), employing “line-by-line” reconstruction similar to our approach, although without explicit accuracy estimates. Similar techniques are applied for functions on a sphere in [19]. A somewhat related approach is taken in [67] utilizing a variational technique with a  $\ell_1$  penalty term, based on the annihilating filter method.

A recent work by D.Levin [50] is most closely related to ours. It employs a similar setting of a discontinuity curve and two smooth pieces, and reconstructs the distance function to the boundary curve  $\Gamma$  by a 2D spline approximation utilizing an iterative method. An initial mesh for approximating the singularity curve is required, and also a global condition for approximating  $\Gamma$  as a zero set of some function from a linear space. The accuracy bounds depend on a certain Lipschitz constant, which is not estimated explicitly.

Several works investigated reconstruction from point-wise data, such as [51, 5], or cell-averages [4, 24], where the piecewise-smooth nature of the function is exploited.

Finally we would like to mention the classical computational harmonic analysis techniques based on sparse representations in overcomplete frames such as wavelets [26], curvelets [21], shearlets [42, 47, 62], wedgelets [29], or adaptive approaches such as bandlets and grouplets [52]. While some of these methods provide optimal reconstruction of cartoon-like images, these typically require the representation coefficients to be available, while also not targeting reconstruction of discontinuity curves directly.

**1.4. Code availability.** The code implementing the algorithms described in this work is available at <https://github.com/mlevinov/algebraic-fourier-2d>.

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**1.6. Organization.** The rest of the manuscript is organized as follows. In section 2 we describe our method in detail and prove the main results. In section 3 we present numerical experiments that demonstrate the accuracy of our method. In section 4 we discuss possible extensions and future work. Some more technical details are deferred to the appendix.

## 2. THE ALGEBRAIC RECONSTRUCTION METHOD

Recall from Lemma 1.1 that  $\psi_{\omega_y}$  is in the general case only piecewise-smooth in  $x$ . Therefore, to obtain an accurate approximation of  $\psi_{\omega_y}$ , we apply a combination of the methods developed in [15, 10] for recovering  $\widetilde{\psi_{\omega_y}}$ . By reconstructing  $\psi_{\omega_y}$  we get an approximation for  $\widehat{F}_x(\omega_y)$  and under the assumption that  $F_x(y)$  is a piecewise-smooth 1D function we apply the method described in [15, 10] once again to recover the jump locations and jump magnitudes of  $F_x$  with sufficient accuracy. Thus we can in effect recover  $F$  *slice by slice*.

The rest of this section is organized as follows. In section 2.1 we describe the decomposition of  $\psi_{\omega_y}$  and provide the full details for its recovery. Then in section 2.2 we present the analysis of the approximation of  $\widetilde{\psi_{\omega_y}}$ , proving theorem 2. Afterwards, we proceed to the second step which is tasked with the approximation of the unknown 2D piecewise-smooth function  $F$ . Similarly to the first step, we will begin by describing the decomposition of each *slice*  $F_x$  (see definition 3), also presenting the needed definitions and assumptions for the reconstruction of  $F_x$ . The accuracy of the second step, i.e. the approximation error  $|\widetilde{F}_x - F_x|$ , is analyzed in section 2.4, proving theorem 1.

The full algorithm for our technique is finally presented in section 2.5.

**2.1. The first stage.** We use the following notation: for  $x_0 \in [-\pi, \pi]$  we define

$$(2.1) \quad V_l(x; x_0) = -\frac{(2\pi)^l}{(l+1)!} B_{l+1}\left(\frac{x-x_0}{2\pi}\right), \quad \xi \leq x \leq \xi + 2\pi,$$

where  $V_l(x)$  is understood to be periodically extended to  $[-\pi, \pi]$  and  $B_l(x)$  is the  $l$ -th Bernoulli polynomial [56]. Since  $\psi_{\omega_y}$  has a jump at  $x_0 = -\pi$ , we denote for simplicity  $V_{\psi,l}(x) := V_l(x; -\pi)$ .

We also use the notation  $c_k(f) := \widehat{f}(k)$  to denote the Fourier coefficients of a 1D function  $f$ .

We begin with a decomposition of  $\psi_{\omega_y}$  as defined in eq. (1.6) into a sum of two functions:

$$(2.2) \quad \psi_{\omega_y}(x) = \gamma_{\omega_y}(x) + \phi_{\omega_y}(x), \quad x \in \mathbb{T},$$

where  $\gamma_{\omega_y} \in C_{\mathbb{T}}^{(d_{\omega_y}+1)}$  and  $\frac{d^l}{dx^l} \gamma_{\omega_y}$  is periodic for  $l = 0, \dots, d_{\omega_y}$  and  $\phi_{\omega_y}(x)$  is a piecewise polynomial of degree  $d_{\omega_y}$  with a discontinuity at  $x = -\pi$  which is uniquely determined by  $\{A_l^\psi(\omega_y)\}_{0 \leq l \leq d_{\omega_y}}$  (recall eq. (1.7)) such that it “absorbs” all the discontinuities of  $\psi_{\omega_y}$  and its derivatives. Eckhoff [31, 32, 30] derives the following explicit representation of  $\phi_{\omega_y}(x)$ :

$$(2.3) \quad \phi_{\omega_y}(x) = \sum_{l=0}^d A_l^\psi(\omega_y) V_{\psi,l}(x).$$

**Proposition 6.** *Let  $\phi_{\omega_y}(x)$  be given by eq. (2.3), then for  $\omega_x \in \mathbb{Z}$ :*

$$c_{\omega_x}(\phi_{\omega_y}) = \begin{cases} 0 & \text{if } \omega_x = 0 \\ \frac{(-1)^{\omega_x}}{2\pi} \sum_{l=0}^d \frac{A_l^\psi(\omega_y)}{(i\omega_x)^{l+1}} & \text{if } \omega_x \neq 0 \end{cases}$$

*Proof.* See appendix A.1. □

From eq. (2.2) we get:

$$(2.4) \quad c_k(\psi_{\omega_y}) = c_k(\gamma_{\omega_y}) + c_k(\phi_{\omega_y}).$$

Eckhoff observed that if  $\gamma_{\omega_y}$  is sufficiently smooth, then the contribution of  $c_k(\gamma_{\omega_y})$  to  $c_k(\psi_{\omega_y})$  becomes negligible, i.e. for large  $\omega_x$  we have

$$(2.5) \quad c_{\omega_x}(\psi_{\omega_y}) \approx c_{\omega_x}(\phi_{\omega_y}) = \frac{(-1)^{\omega_x}}{2\pi} \sum_{l=0}^d \frac{A_l^\psi(\omega_y)}{(i\omega_x)^{l+1}}, \quad |\omega_x| \gg 1.$$

As will be explained below, the last approximate equality (cf. eq. (2.11) below) will be used in order to obtain approximate values for the jump magnitudes  $A_l^\psi(\omega_y)$  from the Fourier coefficients of  $\psi_{\omega_y}$ . Let us denote those by  $\widetilde{A}_l^\psi(\omega_y)$ . Then we further define

$$(2.6) \quad c_{\omega_x}(\widetilde{\phi}_{\omega_y}) := \frac{(-1)^{\omega_x}}{2\pi} \sum_{l=0}^d \frac{\widetilde{A}_l^\psi(\omega_y)}{(i\omega_x)^{l+1}}, \quad |\omega_x| \leq M$$

$$(2.7) \quad c_{\omega_x}(\widetilde{\gamma}_{\omega_y}) := c_{\omega_x}(\psi_{\omega_y}) - c_{\omega_x}(\widetilde{\phi}_{\omega_y}), \quad |\omega_x| \leq M$$

$$(2.8) \quad \widetilde{\gamma}_{\omega_y}(x) := \sum_{|\omega_x| \leq M} c_{\omega_x}(\widetilde{\gamma}_{\omega_y}) e^{ix\omega_x}$$

and take the final approximation:

$$(2.9) \quad \begin{aligned} \widetilde{\psi}_{\omega_y}(x) &:= \widetilde{\gamma}_{\omega_y}(x) + \widetilde{\phi}_{\omega_y}(x) \\ &= \sum_{|\omega_x| \leq M} c_{\omega_x}(\widetilde{\gamma}_{\omega_y}) e^{ix\omega_x} + \sum_{l=0}^d \widetilde{A}_l^\psi(\omega_y) V_{\psi,l}(x). \end{aligned}$$

The recovery of the approximate jump magnitudes of  $\frac{d^l}{dx^l} \psi_{\omega_y}$  for  $0 \leq l \leq d$  at  $x = -\pi$  is performed using the so-called *decimation* [10]. In details, we take  $\omega_y \in \mathbb{Z}$ ,  $d \in \mathbb{N}$ ,  $M \gg 1$  and choose the indices  $\omega_x$  to be evenly distributed across the range  $\{0, 1, \dots, M\}$ . So by denoting

$$(2.10) \quad M_1 := \left\lfloor \frac{M}{(d+1)} \right\rfloor$$

and by eq. (2.5) we solve the following linear system in order to extract  $A_l^\psi(\omega_y)$

$$(2.11) \quad \begin{aligned} c_{\omega_x}(\psi_{\omega_y}) &\approx \frac{(-1)^{\omega_x}}{2\pi} \sum_{l=0}^d \frac{A_l^\psi(\omega_y)}{(i\omega_x)^{l+1}} (= c_{\omega_x}(\phi_{\omega_y})) \\ &\text{where } \omega_x = M_1, 2M_1, \dots, (d+1)M_1. \end{aligned}$$

The following result bounds the error in eq. (2.11).

**Lemma 2.1.** *Let  $\omega_y \in \mathbb{Z}$ ,  $k \in \mathbb{N}$  and let there be given  $(2M + 1) \times (2N + 1)$  Fourier coefficients of function  $F : \mathbb{T}^2 \rightarrow \mathbb{R}$  as in eq. (1.1). Let  $\gamma_{\omega_y} \in C_{\mathbb{T}}^{(d+2)}$  as defined in eq. (2.2). Then there exist constants  $A_{d,M_\xi}$  and  $B_F \geq 0$  (see eq. (A.25) and definition 4) such that:*

$$|c_{\omega_x}(\gamma_{\omega_y})| \leq (2d + 5) \left( B_F + A_{d,M_\xi} A_x |\omega_y|^d \right) |\omega_x|^{-d-2}.$$

*Proof.* Immediately follows from Proposition 15 (formulated and proved in appendix A.2) and from [68, Vol. I, chapter 3, Theorem 13.6].  $\square$

Now, in order to recover  $\left\{ A_l^\psi(\omega_y) \right\}_{l=0}^{d_{\omega_y}}$  from the system in eq. (2.11) we define

$$(2.12) \quad \alpha_l(\omega_y) := i^l A_{d_{\omega_y}-l}^\psi(\omega_y)$$

and

$$(2.13) \quad \begin{aligned} m_{\omega_x}(\omega_y) &:= (-1)^{\omega_x} \sum_{l=0}^d \alpha_l(\omega_y) \omega_x^l \\ &\Leftrightarrow \\ (-1)^{-\omega_x} m_{\omega_x}(\omega_y) &:= \sum_{l=0}^d \alpha_l(\omega_y) \omega_x^l \end{aligned}$$

and by multiplying each side of (2.11) by  $2\pi(i\omega_x)^{d+1}$  we get a new definition:

$$(2.14) \quad \begin{aligned} \widetilde{m_{\omega_x}}(\omega_y) &:= 2\pi(i\omega_x)^{d+1} \cdot \widetilde{c_{\omega_x}}(\psi_{\omega_y}) = m_{\omega_x}(\omega_y) + \delta_{\omega_x}(\omega_y), \\ \omega_x &= M_1, 2M_1, \dots, (d+1)M_1, \end{aligned}$$

where by Lemma 2.1 we have

$$|\delta_{\omega_x}(\omega_y)| \leq 2\pi(2d+5) \left( B_F + A_{d,M_\xi} A_x |\omega_y|^d \right) |\omega_x|^{-1}.$$

We are solving this system for  $\omega_x = M_1, 2M_1, \dots, (d+1)M_1$  which leads us to a linear system of order  $(d+1) \times (d+1)$  described here:

$$(2.15) \quad V_N^d := \begin{bmatrix} 1 & N & N^2 & \dots & N^d \\ 1 & 2N & (2N)^2 & \dots & (2N)^d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & (d+1)N & ((d+1)N)^2 & \dots & ((d+1)N)^d \end{bmatrix}$$

Note that  $V_N^d$  is the Vandermonde matrix on  $\{N, 2N, \dots, (d+1)N\}$  and thus it is nondegenerate for all  $N \gg 1$ .

Presenting the linear system in eq. (2.13):

$$(2.16) \quad \begin{bmatrix} (-1)^{M_1} m(M_1, \omega_y) \\ (-1)^{2M_1} m(2M_1, \omega_y) \\ \vdots \\ (-1)^{(d+1)M_1} m((d+1)M_1, \omega_y) \end{bmatrix} = V_{M_1}^d \cdot \begin{bmatrix} \alpha_0(\omega_y) \\ \alpha_1(\omega_y) \\ \alpha_2(\omega_y) \\ \vdots \\ \alpha_d(\omega_y) \end{bmatrix},$$

the solution contains the jump magnitudes of  $\frac{d^l}{dx^l} \psi_{\omega_y}$  where  $l = 0, \dots, d$ , but unfortunately  $m_{\omega_x}(\omega_y)$  are unknown so we use  $\widetilde{m_{\omega_x}}(\omega_y)$  and solve the perturbed linear system:

$$(2.17) \quad \begin{bmatrix} (-1)^{M_1} \widetilde{m}_{M_1}(\omega_y) \\ (-1)^{2M_1} \widetilde{m}_{2M_1}(\omega_y) \\ \vdots \\ (-1)^{(d+1)M_1} \widetilde{m}_{(d+1)M_1}(\omega_y) \end{bmatrix} = V_{M_1}^d \cdot \begin{bmatrix} \widetilde{\alpha}_0(\omega_y) \\ \widetilde{\alpha}_1(\omega_y) \\ \widetilde{\alpha}_2(\omega_y) \\ \vdots \\ \widetilde{\alpha}_d(\omega_y) \end{bmatrix}$$

to get the approximation for the jump magnitudes,  $\left\{ \widetilde{A}_l^\psi(\omega_y) \right\}_{l=0,\dots,d}$ , at  $x = -\pi$ .

The next and final step is to recover  $\psi_{\omega_y}$  by eq. (2.9):

$$\widetilde{\psi}_{\omega_y}(x) = \sum_{|\omega_x| \leq M} c_{\omega_x}(\widetilde{\gamma}_{\omega_y}) e^{i\omega_x x} + \sum_{l=0}^d \widetilde{A}_l^\psi(\omega_y) V_{\psi,l}(x).$$

**2.2. Proof of theorem 2.** This section is dedicated to proving theorem 2.

The jump location of  $\psi_{\omega_y}$  is known to be at  $x = -\pi$  (or at  $\pi$ ), and we define the jump magnitude of  $\frac{d^l}{dx^l}(\psi_{\omega_y})$  as described in eq. (1.7).

Since we assume that for each  $|\omega_y| \leq M_{\omega_y}$  the jump location is known to be at  $\xi = -\pi$ , the first step would be to analyze the accuracy of the jump magnitudes approximation,  $\widetilde{A}_l^\psi(\omega_y)$ . Towards this goal, we have the following result.

**Lemma 2.2.** *Let  $\omega_y \in \mathbb{Z}$ ,  $N \in \mathbb{N}$ ,  $0 < j \in \mathbb{N}$  and  $m_{\omega_x}(\omega_y)$  as defined in eq. (2.14). Then there exist constants  $A_{d,M_\xi}$  and  $B_F \geq 0$  (see eq. (A.25) and definition 4) such that:*

$$(2.18) \quad |\widetilde{m}_{jN}(\omega_y) - m_{jN}(\omega_y)| \leq 2\pi |jN|^{-1} (2d+5) \left( B_F + A_{d,M_\xi} A_x |\omega_y|^d \right).$$

*Proof.* Immediately follows from eq. (2.14). □

*Proof of theorem 2, first estimate.* Denoting

$$\zeta_j := (-1)^{jN} (\widetilde{m}_{jN}(\omega_y) - m_{jN}(\omega_y)),$$

eq. (2.17) implies that for  $N = M_1$  the error vector satisfies:

$$(2.19) \quad \begin{bmatrix} \widetilde{\alpha}_0(\omega_y) - \alpha_0(\omega_y) \\ \widetilde{\alpha}_1(\omega_y) - \alpha_1(\omega_y) \\ \widetilde{\alpha}_2(\omega_y) - \alpha_2(\omega_y) \\ \vdots \\ \widetilde{\alpha}_d(\omega_y) - \alpha_d(\omega_y) \end{bmatrix} = (V_{M_1}^d)^{-1} \cdot \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_{d+1} \end{bmatrix}.$$

Using Lemma 2.2 we have that:

$$(2.20) \quad \begin{aligned} |\zeta_j| &= |\widetilde{m}_{jM_1}(\omega_y) - m_{jM_1}(\omega_y)| \\ &\leq 2\pi |jM_1|^{-1} \leq 2\pi |jM_1|^{-1} (2d+5) \left( B_F + A_{d,M_\xi} A_x |\omega_y|^d \right) \\ &\stackrel{j \geq 1}{\leq} 2\pi (M_1)^{-1} (2d+5) \left( B_F + A_{d,M_\xi} A_x |\omega_y|^d \right). \end{aligned}$$

Going back to (2.15), we rewrite  $V_{M_1}^d$  in order to estimate  $(V_{M_1}^d)^{-1}$  as follows:

$$(2.21) \quad \begin{aligned} V_{M_1}^d &= \begin{bmatrix} 1 & M_1 & \cdots & (M_1)^d \\ 1 & 2M_1 & \cdots & (2M_1)^d \\ \vdots & \vdots & \vdots & \vdots \\ 1 & (d+1)M_1 & \cdots & ((d+1)M_1)^d \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2^d \\ \vdots & \vdots & \vdots & \vdots \\ 1 & (d+1) & \cdots & (d+1)^d \end{bmatrix} \cdot \text{diag}(1, M_1, \dots, (M_1)^d) \\ &= V_1^d \cdot \text{diag}(1, M_1, \dots, (M_1)^d) \\ &\Rightarrow (V_{M_1}^d)^{-1} = \begin{bmatrix} 1 & & & \\ & (M_1)^{-1} & & \\ & & \ddots & \\ & & & (M_1)^{-d} \end{bmatrix} (V_1^d)^{-1}. \end{aligned}$$



Inserting the above into eq. (2.19) we obtain

$$(2.22) \quad \begin{bmatrix} \widetilde{\alpha}_0(\omega_y) - \alpha_0(\omega_y) \\ \widetilde{\alpha}_1(\omega_y) - \alpha_1(\omega_y) \\ \widetilde{\alpha}_2(\omega_y) - \alpha_2(\omega_y) \\ \vdots \\ \widetilde{\alpha}_d(\omega_y) - \alpha_d(\omega_y) \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ & (M_1)^{-1} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & (M_1)^{-d} \end{bmatrix} (V_1^d)^{-1} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_{d+1} \end{bmatrix}.$$

Presently, we consider the  $l^{\text{th}}$  row of  $(V_1^d)^{-1} = (a_{l,1}, \dots, a_{l,d+1})$ , and obtain

$$\widetilde{\alpha}_l(\omega_y) - \alpha_l(\omega_y) = (M_1)^{-l} \sum_{k=1}^{d+1} a_{l,k} \zeta_k.$$

Denote  $C_l$  as the upper bound on the absolute sum of the entries of the  $l^{\text{th}}$  row of  $(V_1^d)$ :

$$(2.23) \quad \sum_{k=1}^{d+1} |a_{l,k}| \leq C_l \leq C^* := \max_{0 \leq l \leq d+1} \{C_l\}.$$

Using this bound with eq. (2.20) we proceed as follows:

$$\begin{aligned} |\widetilde{\alpha}_l(\omega_y) - \alpha_l(\omega_y)| &= \left| M_1^{-l} \sum_{k=1}^{d+1} a_{l,k} \zeta_k \right| \leq M_1^{-l} \sum_{k=1}^{d+1} |a_{l,k}| |\zeta_k| \\ &\leq M_1^{-l} \sum_{k=1}^{d+1} \frac{2\pi}{M_1} (2d+5) (B_F + A_{d,M_\xi} A_x |\omega_y|^d) |a_{l,k}| \\ &= \frac{2\pi}{(M_1)^{l+1}} (2d+5) (B_F + A_{d,M_\xi} A_x |\omega_y|^d) \sum_{k=1}^{d+1} |a_{l,k}| \\ &\leq 2\pi C_l (2d+5) (B_F + A_{d,M_\xi} A_x |\omega_y|^d) M_1^{-l-1} \\ &\leq \frac{2\pi C_l (2d+5) (B_F + A_{d,M_\xi} A_x |\omega_y|^d)}{\lfloor \frac{M}{d+2} \rfloor^{l+1}}. \end{aligned}$$

Assuming  $\frac{M}{d+2} - 1 > \frac{M}{2(d+2)} \Leftrightarrow M > 2(d+2)$  we further have:

$$\begin{aligned} |\widetilde{\alpha}_l(\omega_y) - \alpha_l(\omega_y)| &\leq 2\pi C_l (2d+4)^{l+1} (2d+5) (B_F + A_{d,M_\xi} A_x |\omega_y|^d) M^{-l-1} \\ &\leq 2\pi C_l (2d+5)^{l+2} (B_F + A_{d,M_\xi} A_x |\omega_y|^d) M^{-l-1}. \end{aligned}$$

Now let us define the constants

$$(2.24) \quad \mathcal{H} := \mathcal{H}(d, B_F, M_\xi) \text{ s.t. } B_F + A_{d,M_\xi} A_x |\omega_y|^d \leq \mathcal{H} A_x |\omega_y|^d,$$

$$(2.25) \quad \mathcal{H}_l := \mathcal{H}_l(d, B_F, M_\xi) = 2\pi C_l (2d+5)^{l+2} \mathcal{H}.$$

This implies

$$|\widetilde{\alpha}_l(\omega_y) - \alpha_l(\omega_y)| \leq \mathcal{H}_l A_x |\omega_y|^d M^{-l-1}$$

and

$$\begin{aligned} \left| \widetilde{\alpha_l(\omega_y)} - \alpha_l(\omega_y) \right| &= \left| i^{d+1-l} \widetilde{A_{d-l}^\psi(\omega_y)} - i^{d+1-l} A_{d-l}^\psi(\omega_y) \right| \\ &= \left| \widetilde{A_{d-l}^\psi(\omega_y)} - A_{d-l}^\psi(\omega_y) \right|. \end{aligned}$$

Finally we have:

$$(2.26) \quad \left| \widetilde{A_{d-l}^\psi(\omega_y)} - A_{d-l}^\psi(\omega_y) \right| \leq \mathcal{H}_l A_x |\omega_y|^d M^{-l-1},$$

for  $l = 0, \dots, d$

and by reindexing, we have

$$(2.27) \quad \left| \widetilde{A}_l^\psi(\omega_y) - A_l^\psi(\omega_y) \right| \leq \mathcal{H}_{d-l} A_x |\omega_y|^d M^{l-d-1},$$

for  $l = 0, \dots, d$ .

completing the proof. □

*Proof of theorem 2, second estimate.* Recall that by eq. (2.2) we have  $\gamma_{\omega_y} = \psi_{\omega_y} - \phi_{\omega_y}$  with  $\gamma_{\omega_y}$  being  $d$ -times everywhere continuously differentiable function. We denote

$$(2.28) \quad \psi_{\omega_y, M}(x) := \sum_{|\omega_x| \leq M} (c_{\omega_x}(\psi_{\omega_y}) - c_{\omega_x}(\phi_{\omega_y})) e^{ix\omega_x} + \phi_{\omega_y}(x).$$

We will estimate the pointwise approximation error of  $\psi_{\omega_y}$  by bounding each term in the decomposition:

$$(2.29) \quad \left| \widetilde{\psi}_{\omega_y}(x) - \psi_{\omega_y}(x) \right| \leq \left| \widetilde{\psi}_{\omega_y}(x) - \psi_{\omega_y, M}(x) \right| + \left| \psi_{\omega_y, M}(x) - \psi_{\omega_y}(x) \right|.$$

We commence by estimating the first term in the right-hand side of eq. (2.29). Starting with a definition:

$$\Theta_d^*(\omega_y) := \widetilde{\phi}_{\omega_y} - \phi_{\omega_y},$$

we have

$$\begin{aligned} \left| \widetilde{\psi}_{\omega_y}(x) - \psi_{\omega_y, M}(x) \right| &= \left| \sum_{|\omega_x| \leq M} (c_{\omega_x}(\phi_{\omega_y}) - c_{\omega_x}(\widetilde{\phi}_{\omega_y})) e^{ix\omega_x} + (\widetilde{\phi}_{\omega_y}(x) - \phi_{\omega_y}(x)) \right| \\ &= \left| \Theta_d^*(\omega_y)(x) - \sum_{|\omega_x| \leq M} c_{\omega_x}(\Theta_d^*(\omega_y)) e^{ix\omega_x} \right| \\ &= \left| \sum_{|\omega_x| > M} c_{\omega_x}(\Theta_d^*(\omega_y)) e^{ix\omega_x} \right| \end{aligned}$$

Using eq. (2.3) we also have

$$\Theta_d^*(\omega_y)(x) = \sum_{l=0}^d \left( \widetilde{A}_l^\psi(\omega_y) - A_l^\psi(\omega_y) \right) V_{\psi, l}(x),$$

thus:

$$\begin{aligned} c_{\omega_x}(\Theta_d^*(\omega_y)) &= c_{\omega_x} \left( \sum_{l=0}^d \left( \widetilde{A}_l^\psi(\omega_y) - A_l^\psi(\omega_y) \right) V_{\psi, l}(x) \right) \\ &= \left( \sum_{l=0}^d \left( \widetilde{A}_l^\psi(\omega_y) - A_l^\psi(\omega_y) \right) c_{\omega_y}(V_{\psi, l}) \right). \end{aligned}$$

Using  $V_{\psi, l} \in C_{\mathbb{T}}^l$  and a well-known estimate (see [41, Section. 3, p. 27]) we claim that there exist positive constants  $\{T_l\}_{l=0}^d$  such that:

$$(2.30) \quad \left| \sum_{|\omega_x| > M} c_{\omega_x}(V_{\psi, l}) e^{ix\omega_x} \right| \leq T_l M^{-l} \leq T^* M^{-l}$$

where  $T^* := \max_{0 \leq l \leq d} \{T_l\}$  and therefore together with eq. (2.25) we get:

$$\begin{aligned}
\left| \sum_{|\omega_x| > M} c_{\omega_x}(\Theta_d^*(\omega_y)) e^{ix\omega_x} \right| &\leq 2\pi T_l C_{d-l} (2d+5) \left( B_F + A_{d, M_\xi} A_x |\omega_y|^d \right) (M_1)^{l-d-1} M^{-l} \\
&\leq \frac{2\pi T_l C_{d-l} (2d+5) \left( B_F + A_{d, M_\xi} A_x |\omega_y|^d \right)}{(M_1)^{d+1-l} M^l} \\
&\leq \frac{2\pi T_l C_{d-l} (2d+5) \left( B_F + A_{d, M_\xi} A_x |\omega_y|^d \right)}{\left( \frac{M}{d+2} - 1 \right)^{d+1-l} M^l} \\
&\leq \frac{2\pi T_l C_{d-l} (2d+5) \left( B_F + A_{d, M_\xi} A_x |\omega_y|^d \right)}{\left( \frac{M}{2(d+2)} \right)^{d+1-l} M^l} \\
&= \left( 2\pi T_l C_{d-l} (2d+5) \left( B_F + A_{d, M_\xi} A_x |\omega_y|^d \right) (2d+4)^{d+1-l} \right) M^{-d-1} \\
&\leq \left( 2\pi T_l C_{d-l} \left( B_F + A_{d, M_\xi} A_x |\omega_y|^d \right) (2d+5)^{d+2-l} \right) M^{-d-1} \\
&\leq T_l \mathcal{H}_{d-l} A_x |\omega_y|^d M^{-d-1} \leq T^* \mathcal{H}_{d-l} A_x |\omega_y|^d M^{-d-1}.
\end{aligned}$$

Denoting

$$(2.31) \quad \mathcal{H}_T^* := T^* \cdot \max_{0 \leq l \leq d} \{ \mathcal{H}_l \}$$

we get

$$(2.32) \quad \left| \widetilde{\psi}_{\omega_y}(x) - \psi_{\omega_y, M}(x) \right| \leq \mathcal{H}_T^* A_x |\omega_y|^d M^{-d-1}.$$

Now we estimate the second term in the right-hand side of eq. (2.29). Combining eq. (2.2) and eq. (2.28) we have:

$$(2.33) \quad \left| \psi_{\omega_y, M}(x) - \psi_{\omega_y}(x) \right| = \left| \sum_{|\omega_x| > M} c_{\omega_x}(\gamma_{\omega_y}) e^{ix\omega_x} \right|.$$

Using the same analysis as in [15, p.301] together with Lemma 2.1 and eq. (2.2) implies that there exists a constant  $R$  s.t.:

$$(2.34) \quad \begin{aligned} \left| \psi_{\omega_y, M}(x) - \psi_{\omega_y}(x) \right| &\leq R(2d+5) \left( B_F + A_{d, M_\xi} A_x |\omega_y|^d \right) M^{-d-1} \\ &\leq R(2d+5) \mathcal{H} A_x M^{-d-1}. \end{aligned}$$

Combining eqs. (2.32) and (2.33) we get:

$$\begin{aligned}
\left| \widetilde{\psi}_{\omega_y}(x) - \psi_{\omega_y, M}(x) \right| &\leq \mathcal{H}_T^* A_x |\omega_y|^d M^{-d-1} = \\
&= (\mathcal{H}_T^* + R(2d+5) \mathcal{H}) A_x |\omega_y|^d M^{-d-1} + R(2d+5) \mathcal{H} A_x M^{-d-1}.
\end{aligned}$$

Denoting:

$$(2.35) \quad R_d := (2d+5)R$$

we can conclude that for  $\omega_y \in \mathbb{Z}$  and  $x \in \mathbb{T}$  we have:

$$(2.36) \quad \left| \widetilde{\psi}_{\omega_y}(x) - \psi_{\omega_y}(x) \right| \leq (\mathcal{H}_T^* + R_d \mathcal{H}) A_x |\omega_y|^d M^{-d-1}.$$

This completes the proof of theorem 2. □

**2.3. The second stage.** So far, we have described the reconstruction process of  $\{\widetilde{\psi_{\omega_y}}(x)\}_{|\omega_y| \leq N}$  for a given  $x \in \mathbb{T}$  by applying algorithm 2 to the given data  $\{\widehat{F}(\omega_x, \omega_y)\}_{|\omega_x| \leq M}$  for each  $|\omega_y| \leq N$ . In the second stage, we will use these approximated values of  $\widehat{F}_x(\omega_y)$  (recall that  $\psi_{\omega_y}(x) = \widehat{F}_x(\omega_y)$ ) to recover  $F_x(y)$  for each  $y \in \mathbb{T}$ . This way, we eventually recover  $F_x(y)$  for each  $x, y \in \mathbb{T}$ , and thus recover  $F$  itself. (In practice, this can be done for a finite number of  $x$  and  $y$ ). Now we present the decomposition for a slice of  $F$  at  $x \in \mathbb{T}$ :

**Proposition 7.** *Let  $x \in \mathbb{T}$ , then*

$$(2.37) \quad F_x(y) = \Gamma_x(y) + \Phi_{d,x}(y), \quad \forall y \in \mathbb{T},$$

where  $\Gamma_x \in C_{\mathbb{T}}^{(d+1)}$ , and  $\Phi_{d,x}$  is a piecewise polynomial of degree  $d$  “absorbing” all the discontinuities of  $F_x$  and its  $d+1$  derivatives.

In particular,  $\Phi_{d,x}$  has a single discontinuity at  $y_x = \xi(x)$  and therefore it is uniquely determined by  $\{A_l(x)\}_{0 \leq l \leq d}$  and  $y_x$  as follows:

$$(2.38) \quad \Phi_{d,x}(y) = \sum_{l=0}^d A_l(x) V_l(y; y_x).$$

Here, again,  $V_l(y; y_x)$  is the periodic Bernoulli polynomial, as in eq. (2.1). Furthermore, we have that  $\Gamma_x \in C_{\mathbb{T}}^{(d+1)}$  which in turn by [68, Vol. I, Chapter 3, Theorem 13.6] gives us, for  $x \in \mathbb{T}$ , a constant  $R_{\Gamma_x}$  s.t.:

$$(2.39) \quad |c_{\omega_y}(\Gamma_x)| \leq R_{\Gamma_x} |\omega_y|^{-d-2}.$$

We assume a uniform bound over the Fourier coefficients of  $\Gamma_x$  as follows:

$$(2.40) \quad |c_{\omega_y}(\Gamma_x)| \leq R_{\Gamma_x} |\omega_y|^{-d-2} \leq R_{\Gamma_{\mathbb{T}}} |\omega_y|^{-d-2}, \quad \forall x \in \mathbb{T}.$$

Now we denote:

$$(2.41) \quad \kappa_x := e^{-i\xi(x)}$$

and get:

**Proposition 8.** *Let  $x \in \mathbb{T}$  and  $\Phi_{d,x}$  as given by (2.38), then for  $\omega_y \in \mathbb{Z}$ :*

$$(2.42) \quad c_{\omega_y}(\Phi_{d,x}) = \begin{cases} 0 & \text{if } \omega_y = 0, \\ \frac{(\kappa_x)^{\omega_x}}{2\pi} \sum_{l=0}^d \frac{A_l(x)}{(i\omega_y)^{(l+1)}} & \text{otherwise.} \end{cases}$$

*Proof.* Repeat the proof for Proposition 6. □

Using eq. (1.6) for each  $x \in \mathbb{T}$  and  $\omega_y \in \mathbb{Z}$  we denote:

$$(2.43) \quad c_{\omega_y}(F_x) := \psi_{\omega_y}(x)$$

and from eq. (2.36) we get:

$$(2.44) \quad \begin{aligned} \widetilde{\psi_{\omega_y}}(x) &:= \psi_{\omega_y}(x) + \delta_{\omega_y}(x), \quad \text{where} \\ |\delta_{\omega_y}(x)| &\leq (\mathcal{H}_T^* + R_d \mathcal{H}) A_x |\omega_y|^d M^{-d-1}. \end{aligned}$$

In order to recover the parameters  $y_x = \xi(x)$  and  $\{A_l(x)\}_{l=0}^d$  of  $F_x$  and to approximate  $F_x(y)$  at  $y \in \mathbb{T}$  we would ideally need to have access to  $\{c_{\omega_y}(F_x)\}$  for  $|\omega_y| \leq N$  which are not available. Instead we use the approximation of  $c_{\omega_y}(F_x)$  which is given in eq. (2.44) by  $\widetilde{\psi_{\omega_y}}(x)$ . To apply decimation once again, we denote:

$$N_1 := \left\lfloor \frac{N}{d+2} \right\rfloor.$$

Using Proposition 7 we further denote:

$$(2.45) \quad c_{\omega_y}(F_x) := c_{\omega_y}(\Gamma_x) + c_{\omega_y}(\Phi_{d,x})$$

and by eqs. (2.43) and (2.44) we have:

$$c_{\omega_y}(F_x) = \psi_{\omega_y}(x) = c_{\omega_y}(\Gamma_x) + c_{\omega_y}(\Phi_{d,x})$$

which gives us:

$$c_{\omega_y}(F_x) = \widetilde{\psi}_{\omega_y}(x) - \delta_{\omega_y}(x) = c_{\omega_y}(\Gamma_x) + c_{\omega_y}(\Phi_{d,x}),$$

therefore

$$(2.46) \quad \widetilde{\psi}_{\omega_y}(x) = c_{\omega_y}(\Gamma_x) + c_{\omega_y}(\Phi_{d,x}) + \delta_{\omega_y}(x)$$

where  $\delta_{\omega_y}(x)$  is given by eq. (2.44)

Now for every  $x \in \mathbb{T}$  we denote:

$$(2.47) \quad \alpha_l(x) = \iota^l A_{d-l}(x)$$

$$(2.48) \quad m_{\omega_y}(x) := \kappa_x^{\omega_y} \sum_{l=0}^d \alpha_l(x) \omega_y^l$$

Multiplying both sides of eq. (2.46) by  $2\pi(\iota\omega_y)^{d+1}$  and using eqs. (2.47) and (2.48) together with Proposition 8 we get:

$$\begin{aligned} 2\pi(\iota\omega_y)^{d+1} \widetilde{\psi}_{\omega_y}(x) &= 2\pi(\iota\omega_y)^{d+1} c_{\omega_y}(\Phi_{d,x}) + 2\pi(\iota\omega_y)^{d+1} (c_{\omega_y}(\Gamma_x) + \delta_{\omega_y}(x)) \\ &= \kappa_x^{\omega_y} \sum_{l=0}^d (\iota\omega_y)^{d-l} A_l(x_0) + 2\pi(\iota\omega_y)^{d+1} (c_{\omega_y}(\Gamma_x) + \delta_{\omega_y}(x)) \\ &= m_{\omega_y}(x) + 2\pi(\iota\omega_y)^{d+1} (c_{\omega_y}(\Gamma_x) + \delta_{\omega_y}(x)). \end{aligned}$$

Next we define:

$$(2.49) \quad \widetilde{m}_{\omega_y}(x) := m_{\omega_y}(x) + \varepsilon_{\omega_y}(x)$$

where  $\varepsilon_{\omega_y}(x) = 2\pi(\iota\omega_y)^{d+1} (c_{\omega_y}(\Gamma_x) + \delta_{\omega_y}(x))$  and:

$$\begin{aligned} (2.50) \quad |\varepsilon_{\omega_y}(x)| &= 2\pi|\omega_y|^{d+1} |c_{\omega_y}(\Gamma_x) + \delta_{\omega_y}(x)| \\ &\leq 2\pi|\omega_y|^{d+1} (|c_{\omega_y}(\Gamma_x)| + |\delta_{\omega_y}(x)|) \\ &\text{using eqs. (2.40) and (2.44) we continue} \\ &\leq 2\pi \left( R_{\Gamma_{\mathbb{T}}} |\omega_y|^{-1} + (\mathcal{H}_T^* + R_d \mathcal{H}) A_x |\omega_y|^{2d+1} M^{-d-1} \right). \end{aligned}$$

The decimated system in eq. (2.48) is solved in two steps. First, a polynomial equation  $q_{N_1}^d(u) = 0$  is constructed from the perturbed coefficients  $\{\widetilde{\psi}_{\omega_y}(x)\}_{\omega_y=N_1, 2N_1, \dots, (d+2)N_1}$  and denoted by:

$$(2.51) \quad q_{N_1}^d(u) = \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} \widetilde{m}_{(j+1)N_1}(x) u^{d+1-j}.$$

**Definition 9.** Let  $x \in \mathbb{T}$ , then

$$(2.52) \quad p_{N_1}^d(u) := \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} m_{(j+1)N_1}(x) u^{d+1-j}.$$

Therefore,  $q_{N_1}^d$  is a perturbation of  $p_{N_1}^d(u)$  which is constructed from the unperturbed and unknown values of  $\{c_{\omega_y}(F_x)\}$ .

**Proposition 10.** The point  $u = \kappa_x^{N_1} = e^{-\iota\xi(x)N_1}$  is a root of  $p_{N_1}^d$ .

*Proof.* Denoting  $z = \kappa_x^{N_1}$  we get:

$$\begin{aligned}
p_{N_1}^d(z) &= \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} m_{(j+1)N_1}(x) z^{d+1-j} \\
&= \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} \left( z^{j+1} \sum_{l=0}^d \alpha_l(x) (j+1) N_1^l \right) z^{d+1-j} \\
&= \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} z^{d+2} \left( \sum_{l=0}^d \alpha_l(x) (j+1)^l N_1^l \right) \\
&= z^{d+2} \sum_{l=0}^d \alpha_l(x) N_1^l \left\{ \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} (j+1)^l \right\}
\end{aligned}$$

Now we notice that if we apply the forward difference operator  $d+1$  times over the polynomial  $f(x) = x^l$  we get:

$$\delta^{d+1} f(x) = \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} (j+1)^l$$

but because  $l < d+1$  using [20, p. 6] we conclude that the above equals 0, implying that  $p_{N_1}^d(u) = 0$ .  $\square$

Now we understand that one of the roots of  $p_{N_1}^d(u)$  is  $u = \kappa_x^{N_1}$  and thus, by solving the perturbed equation  $q_{N_1}^d(u) = 0$  we will recover the approximation of  $\kappa_x^{N_1}$ , which is  $\widetilde{\kappa}_x^{N_1} = e^{-i\xi(x)N_1}$  and by extracting the  $N_1^{th}$  root and subsequently taking logarithm we obtain the approximation of the jump location of  $F_x$ , which is  $\widetilde{y}_x = \widetilde{\xi}(x)$ . The operation of taking root generally results in a multi-valued solution, therefore to ensure correct reconstruction, we additionally assume that the jump  $\xi(x)$  must be known with *a priori* accuracy of order  $o((N_1)^{-1})$ , which is valid because it is always possible to apply the half order algorithm such as in [10, p. 4] in order to achieve the approximation of the jump location with the assumed accuracy.

Now the second step is to recover the jump magnitudes  $\{A_l(x)\}_{l=0}^d$  by solving the linear system of equations in Proposition 12 using the approximated jump location  $\widetilde{\kappa}_x$ .

**Definition 11.** Let  $V_N^d$  denote the  $(d+1) \times (d+1)$  matrix

$$(2.53) \quad V_N^d := \begin{bmatrix} 1 & N & N^2 & \dots & N^d \\ 1 & 2N & (2N)^2 & \dots & (2N)^d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & (d+1)N & ((d+1)N)^2 & \dots & ((d+1)N)^d \end{bmatrix}$$

Note that  $V_N^d$  is the Vandermonde matrix on  $\{N, 2N, \dots, (d+1)N\}$  and thus it is nondegenerate for all  $0 \neq N \in \mathbb{N}$ .

**Proposition 12.** The vector of exact magnitudes  $(\alpha_0(x), \dots, \alpha_d(x))^T$  satisfies:

$$(2.54) \quad \begin{bmatrix} m_{N_1}(x) \cdot (\kappa_x)^{-N_1} \\ m_{2N_1}(x) \cdot (\kappa_x)^{-2N_1} \\ \vdots \\ m_{(d+1)N_1}(x) \cdot (\kappa_x)^{-(d+1)N_1} \end{bmatrix} = V_{N_1}^d \cdot \begin{bmatrix} \alpha_0(x) \\ \alpha_1(x) \\ \vdots \\ \alpha_d(x) \end{bmatrix}.$$

*Proof.* Immediately follows from eq. (2.48)  $\square$

The solution contains the jump magnitudes of  $\frac{d^l}{dx^l} F_x$  where  $l = 0, \dots, d$ , but unfortunately  $m_{\omega_y}(x)$  isn't known so we use  $\widetilde{m}_{\omega_y}(x)$  and solve the perturbed linear system:

$$(2.55) \quad \begin{bmatrix} \widetilde{m}_{N_1}(x) \widetilde{\kappa}_x^{-N_1} \\ \widetilde{m}_{2N_1}(x) \widetilde{\kappa}_x^{-2N_1} \\ \vdots \\ \widetilde{m}_{(d+1)N_1}(x) \widetilde{\kappa}_x^{-(d+1)N_1} \end{bmatrix} = V_{N_1}^d \cdot \begin{bmatrix} \widetilde{\alpha}_0(x) \\ \widetilde{\alpha}_1(x) \\ \vdots \\ \widetilde{\alpha}_d(x) \end{bmatrix},$$

to get the approximation for the jump magnitudes,  $\left\{\widetilde{A}_l(x)\right\}_{l=0}^d$ , where  $\widetilde{\alpha}_l(x) = \iota^l \widetilde{A}_{d-l}(x)$ .

The next and final step is to recover  $F_x$  by:

$$\begin{aligned}
(2.56) \quad \widetilde{F}_x(y) &= \sum_{|\omega_y| \leq N} c_{\omega_y}(\widetilde{\Gamma}_x) e^{\iota \omega_y y} + \sum_{l=0}^d \widetilde{A}_l(x_0) V_l(y; \widetilde{y}_x) \\
&= \sum_{|\omega_y| \leq N} \left( \widetilde{\psi}_{\omega_y}(x) - c_{\omega_y}(\widetilde{\Phi}_{d,x}) \right) e^{\iota \omega_y y} + \sum_{l=0}^d \widetilde{A}_l(x_0) V_l(y; \widetilde{y}_x).
\end{aligned}$$

**2.4. Proof of theorem 1.** Here we present the proof for theorem 1. In contrast to section 2.2 the jump location is unknown and we begin our calculations with the error in approximating the jump location. The second step would be to analyze the error in approximating the jump magnitudes of  $F_x$  up to the chosen reconstruction order of  $F_x$ ; in the third (and last) step we incorporate the former two steps and analyze the error of a pointwise approximation of  $F_x$ .

**Lemma 2.3.** *Let  $x \in \mathbb{T}$  and  $F_x$  as in definition 3,  $F_x \in PC_{\mathbb{T}}^{(d+1,1)}$  and let  $\psi_{\omega_y} \in PC_{\mathbb{T}}^{(d+1,1)}$  as in eq. (1.6) and Lemma 1.1. Assume that*

$$N^2 \leq M.$$

Then there exists  $\mathcal{R}_{\Gamma|\gamma} := \mathcal{R}_{\Gamma|\gamma}(d, B_F, M_\xi, R_{\Gamma_T}, R)$  as defined in eq. (2.58) below such that:

$$(2.57) \quad |\widetilde{m}_{\omega_y}(x) - m_{\omega_y}(x)| \leq \mathcal{R}_{\Gamma|\gamma} A_x |\omega_y|^{-1}.$$

*Proof.* Using eq. (2.50) we write:

$$\begin{aligned}
|\widetilde{m}_{\omega_y}(x) - m_{\omega_y}(x)| &\leq 2\pi \left( R_{\Gamma_T} |\omega_y|^{-1} + (\mathcal{H}_T^* + R_d \mathcal{H}) A_x |\omega_y|^{2d+1} M^{-d-1} \right) \\
&= 2\pi \left( \frac{R_{\Gamma_T}}{|\omega_y|} + \frac{(\mathcal{H}_T^* + R_d \mathcal{H}) A_x}{|\omega_y|} \cdot \frac{|\omega_y|^{2d+2}}{M^{d+1}} \right) \\
|\omega_y| \leq N &\leq 2\pi \left( \frac{R_{\Gamma_T}}{|\omega_y|} + \frac{(\mathcal{H}_T^* + R_d \mathcal{H}) A_x}{|\omega_y|} \cdot \frac{N^{2d+2}}{M^{d+1}} \right) \\
&= 2\pi \left( \frac{R_{\Gamma_T}}{|\omega_y|} + \frac{(\mathcal{H}_T^* + R_d \mathcal{H}) A_x}{|\omega_y|} \cdot \left( \frac{N^2}{M} \right)^{d+1} \right) \\
N^2 \leq M &\leq 2\pi \left( \frac{R_{\Gamma_T} + (\mathcal{H}_T^* + R_d \mathcal{H}) A_x}{|\omega_y|} \right)
\end{aligned}$$

and we get:

$$|\widetilde{m}_{\omega_y}(x) - m_{\omega_y}(x)| \leq 2\pi (R_{\Gamma_T} + (\mathcal{H}_T^* + R_d \mathcal{H}) A_x) |\omega_y|^{-1}.$$

Because  $R_{\Gamma_T}$ ,  $R_d$ ,  $\mathcal{H}_T^*$  and  $\mathcal{H}$  are constants which are dependent only on  $d$ ,  $B_F$ ,  $M_\xi$  and  $R$  (recall eq. (2.34)) we will define a function that will replace this complicated structure for the sake of clarity in the following way:

$$\begin{aligned}
(2.58) \quad \mathcal{R}_{\Gamma|\gamma} &:= \mathcal{R}_{\Gamma|\gamma}(d, B_F, M_\xi, R_{\Gamma_T}, R) \text{ s.t.:} \\
2\pi (R_{\Gamma_T} + (\mathcal{H}_T^* + R_d \mathcal{H}) A_x) &\leq \mathcal{R}_{\Gamma|\gamma} A_x
\end{aligned}$$

and therefore we can finish the proof:

$$|\widetilde{m}_{\omega_y}(x) - m_{\omega_y}(x)| \leq \mathcal{R}_{\Gamma|\gamma} A_x |\omega_y|^{-1}. \quad \square$$

**2.4.1. Jump location.** Recall definition 4. The main approximation result for the jump location is the following.

**Proposition 13.** *Let  $N_1 := \left\lfloor \frac{N}{d+2} \right\rfloor$ ,  $p_{N_1}^d$  and  $q_{N_1}^d$  be given by definition 9 and eq. (2.51), respectively. Then there exists  $C_{2,d}$  and  $\mathcal{R}_{\Gamma|\gamma}$  (see eqs. (2.58) and (2.61)) such that*

$$(2.59) \quad |\widetilde{y}_x - y_x| \leq C_{2,d} \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{-d-2}.$$

*Proof.* Let  $\{y_1^{(N_1)}, \dots, y_d^{(N_1)}\}$  be the roots of  $q_{N_1}^d$  and  $\{u_1^{(N_1)}, \dots, u_d^{(N_1)}\}$  be the roots of  $p_{N_1}^d$ . Using Lemma 18 from [10] and Lemma 2.3 implies that for a large  $N_1 = \lfloor \frac{N}{d+2} \rfloor$  there exists  $C_{1,d} := C_1(d)$  such that:

$$\left| y_i^{(N_1)} - u_i^{(N_1)} \right| \leq C_{1,d} \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma}(N_1)^{-d-1}.$$

Keeping in mind that one of the roots of  $p_{N_1}^d$  is  $\kappa_x^{N_1} = (e^{-iy_x})^{N_1}$ , so by using the bound found above we write:

$$\widetilde{\kappa}_x^{N_1} = \kappa_x^{N_1} + \frac{C_1^*(N_1)}{(N_1)^{d+1}}, \text{ where } |C_1^*(N_1)| \leq C_{1,d} \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma}.$$

Extraction of the  $(N_1)^{th}$  root further decreases the error by the factor of  $\frac{1}{N_1}$ :

$$\begin{aligned} |\widetilde{\kappa}_x - \kappa_x| &= \left| 1 - \frac{\widetilde{\kappa}_x}{\kappa_x} \right| = \left| 1 - \left( \frac{\widetilde{\kappa}_x^{N_1}}{\kappa_x^{N_1}} \right)^{\frac{1}{N_1}} \right| \\ &= \left| 1 - \left( 1 + \frac{C_1^{**}(N_1)}{(N_1)^{d+1}} \right)^{\frac{1}{N_1}} \right| \\ &\stackrel{\text{Bernoulli's inequality}}{\leq} C_{1,d} \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma}(N_1)^{-d-2}. \end{aligned}$$

The final step would be to recover  $y_x$  from  $\kappa_x = e^{-iy_x}$ . Using the above bound we write  $\widetilde{\kappa}_x = \kappa_x + C_1^\#(N_1) \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma}(N_1)^{-d-2}$ ,  $|C_1^\#(N)| \leq C_{1,d}$  and get:

$$\begin{aligned} |\widetilde{y}_{x_0} - y_{x_0}| &= \left| \log \left( \frac{\widetilde{y}_{x_0}}{y_{x_0}} \right) \right| = \left| \log \left( 1 + C_1^{\#\#}(N_1) \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma}(N_1)^{-d-2} \right) \right|, \\ &\text{where } |C_1^{\#\#}(N_1)| \leq C_{1,d}. \end{aligned}$$

Using the estimation of  $|\log(1 + \varepsilon)| < 2|\varepsilon|$  for  $|\varepsilon| \ll 1$  we get:

$$|\widetilde{y}_x - y_x| \leq 2C_{1,d} \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N_1^{-d-2}.$$

Since  $N_1 = \lfloor \frac{N}{d+2} \rfloor$  and by assuming that  $N > 2(d+2)$  we have that for  $k \in \mathbb{N}$ :

$$(2.60) \quad N^{-k} \leq (2(d+2))^k N^{-k}.$$

Denoting:

$$C_{2,d} := 2C_{1,d} (2d+4)^{d+2},$$

we finally have:

$$(2.61) \quad |\widetilde{y}_x - y_x| \leq C_{2,d} \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{-d-2}$$

completing the proof of Proposition 13.  $\square$

**2.4.2. Jump magnitudes.** Now we proceed to the second part of theorem 1 and investigate the accuracy of recovering the jump magnitudes of  $\frac{d^l}{dx^l} F_x$  at  $y_x$ . Subtracting the unperturbed system eq. (2.54) from the perturbed system eq. (2.55) we get that the error vector satisfies:

$$(2.62) \quad \begin{bmatrix} \widetilde{\alpha}_0(x) - \alpha_0(x_0) \\ \widetilde{\alpha}_1(x) - \alpha_1(x_0) \\ \vdots \\ \widetilde{\alpha}_d(x) - \alpha_d(x_0) \end{bmatrix} = (V_{N_1}^d)^{-1} \begin{bmatrix} \widetilde{m}_{N_1}(x) \widetilde{\kappa}_x^{-N_1} - m_{x_0}(N_1) \kappa_x^{-N_1} \\ \widetilde{m}_{2N_1}(x) \widetilde{\kappa}_x^{-2N_1} - m_{x_0}(2N_1) \kappa_x^{-2N_1} \\ \vdots \\ \widetilde{m}_{(d+1)N_1}(x) \widetilde{\kappa}_x^{-(d+1)N_1} - m_{x_0}((d+1)N_1) \kappa_x^{-(d+1)N_1} \end{bmatrix}.$$



Going back to the bound  $A_x$  from definition 4 and to eq. (2.48) we have:

$$\begin{aligned}
|m_{(j+1)N_1}(x)| &= \left| \kappa_x^{(j+1)N_1} \sum_{l=0}^d \alpha_l(x) (j+1)^l N_1^l \right| \\
&\leq \left| \kappa_x^{(j+1)N_1} \sum_{l=0}^d |\alpha_l(x)| (j+1)^l N_1^l \right| \\
&= \sum_{l=0}^d |\alpha_l(x)| (j+1)^l N_1^l \leq A_x \sum_{l=0}^d (j+1)^l N_1^l \leq A_x \sum_{l=0}^d (j+1)^d N_1^d \\
&= A_x (d+1) (j+1)^d N_1^d \\
&\leq A_x (d+1) (d+1)^d N_1^d = A_x (d+1)^{d+1} N_1^d.
\end{aligned}$$

Denoting  $C_{3,d} = (d+1)^{d+1}$  we get:

$$|m_{(j+1)N_1}(x)| \leq C_{3,d} A_x N_1^d.$$

Applying Lemma 2.3 we have that there exists  $\mathcal{R}_{\Gamma|\gamma}(N_1)$  which depends only on  $N_1$  such that:

$$(2.63) \quad \tilde{m}_{(j+1)N_1}(x) = m_{(j+1)N_1}(x) + \mathcal{R}_{\Gamma|\gamma}(N_1) N_1^{-1}, \quad \mathcal{R}_{\Gamma|\gamma}(N_1) \leq \mathcal{R}_{\Gamma|\gamma}$$

and from the proof of Proposition 13 there exists  $C_1^\#(N_1)$ , which depends only on  $N_1$  such that:

$$\tilde{\kappa}_x = \kappa_x + C_1^\#(N_1) \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N_1^{-d-2}, \quad C_1^\#(N_1) \leq C_{2,d}.$$

Applying Taylor majorization, e.g. [15, Proposition A.7], gives the following estimate:

$$\begin{aligned}
\tilde{\kappa}_x^{-(j+1)N_1} &= \left( \kappa_x + C_1^\#(N_1) \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N_1^{-d-2} \right)^{-(j+1)N_1} \\
&= \kappa_x^{-(j+1)N_1} \left( 1 + \frac{C_1^\#(N_1)}{\kappa_x} \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N_1^{-d-2} \right)^{-(j+1)N_1} \\
&= \kappa_x^{-(j+1)N_1} \left( 1 - C_1^{\#\#}(N_1) \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N_1^{-d-1} \right), \quad |C_1^{\#\#}(N_1)| \leq C_{2,d}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
(2.64) \quad &\left| \tilde{m}_{(j+1)N_1}(x) \tilde{\kappa}_x^{-(j+1)N_1} - m_{(j+1)N_1}(x) \kappa_x^{-(j+1)N_1} \right| = \\
&= \left| (m_{(j+1)N_1}(x) + \mathcal{R}_{\Gamma|\gamma}(N_1) N_1^{-1}) \kappa_x^{-(j+1)N_1} \right. \\
&\quad \left. \left( 1 - C_1^{\#\#}(N_1) \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N_1^{-d-1} \right) - m_{(j+1)N_1}(x) \kappa_x^{-(j+1)N_1} \right| \\
&\leq \frac{\mathcal{R}_{\Gamma|\gamma}}{N_1} \left| \frac{C_{2,d} C_{3,d} A_x}{|A_0(x)|} + 1 \right| + O(N_1^{-d-2}) \\
&\leq C_{4,d} \left( 1 + \frac{A_x}{|A_0(x)|} \right) \mathcal{R}_{\Gamma|\gamma} N_1^{-1}, \quad C_{4,d} = \max\{1, C_{2,d}, C_{3,d}\}.
\end{aligned}$$

Denote  $\zeta_j(x) = \tilde{m}_{(j+1)N_1}(x) \tilde{\kappa}_x^{-(j+1)N_1} - m_{(j+1)N_1}(x) \kappa_x^{-(j+1)N_1}$  and let  $v^d_l$  be an upper bound on the sum of absolute values of the entries in the  $l^{th}$  row of  $(V_1^d)^{-1}$ . We obtain:

$$\begin{aligned}
V_{N_1}^d &= V_1^d \cdot \text{diag}\{1, N_1, \dots, N_1^d\} \\
&\Leftrightarrow \\
(V_{N_1}^d)^{-1} &= \text{diag}\{1, N_1^{-1}, \dots, N_1^{-d}\} \cdot (V_1^d)^{-1},
\end{aligned}$$

therefore,

$$(2.65) \quad \begin{bmatrix} \tilde{\alpha}_0(x) - \alpha_0(x) \\ \tilde{\alpha}_1(x) - \alpha_1(x) \\ \vdots \\ \tilde{\alpha}_d(x) - \alpha_d(x) \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & N_1^{-1} & & \\ & & \ddots & \\ & & & N_1^{-d} \end{bmatrix} \cdot (V_1^d)^{-1} \cdot \begin{bmatrix} \zeta_0(x) \\ \zeta_1(x) \\ \vdots \\ \zeta_d(x) \end{bmatrix}.$$

Denoting  $C_{5,d,l} := C_{4,d} \cdot v^d_l$  and applying eq. (2.64) we have, for  $l = 0, \dots, d$ :

$$(2.66) \quad \begin{aligned} |\tilde{\alpha}_l(x) - \alpha_l(x)| &\leq C_{5,d,l} \left(1 + \frac{A_x}{|A_0(x)|}\right) \mathcal{R}_{\Gamma|\gamma} N_1^{-l-1} \\ &\stackrel{A_0(x) \leq A_x}{\leq} 2C_{5,d,l} \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N_1^{-l-1} \end{aligned}$$

and by going back to  $\alpha_l(x) = v^l A_{d-l}(x)$  (see eq. (2.47)) we have that:

$$|\tilde{\alpha}_l(x) - \alpha_l(x)| = \left| \widetilde{A_{d-l}}(x) - A_{d-l}(x) \right|.$$

Finally by denoting

$$C_{6,d-l} := C_{5,d,l} \cdot 2^{l-d} \cdot (d+1)^{l-d-1}$$

and applying eq. (2.60) we conclude that:

$$(2.67) \quad \left| \widetilde{A}_l(x) - A_l(x) \right| \leq C_{6,d-l} \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{l-d-1}.$$

**2.4.3. Pointwise values.** Here we prove the last part of theorem 1, i.e. the accuracy of recovering  $F_x$ .

We begin by denoting

$$(2.68) \quad F_{x,N}(y) := \sum_{|\omega_y| \leq N} (c_{\omega_y}(F_x) - c_{\omega_y}(\Phi_{d,x})) e^{i\omega_y y} + \Phi_{d,x}(y).$$

**Proposition 14.** *Let  $F_x : \mathbb{T} \rightarrow \mathbb{R}$  as described in definition 4 and let  $r > 0$ . Then for a large enough  $M$  and for every  $|\omega_y| \leq N$  for which algorithm 2 recovers  $\{\widetilde{\psi}_{\omega_y}(x)\}$ , as  $\omega_y = -N, \dots, N$  (where also  $N^2 \leq M$ ) with the accuracy as in theorem 2, there exist  $\mathcal{R}_{\Gamma|\gamma}$ ,  $R_{\Gamma_{\mathbb{T}}}$  and  $C_{10,d}$  (see eqs. (2.40), (2.58) and (2.81)) such that:*

$$\left| \widetilde{F}_x(y) - F_x(y) \right| \leq C_{10,d} \frac{A_x(1+A_x)}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{-d-1}, \quad y \in \mathbb{T} \setminus B_r(y_x)$$

where  $B_r(y_x)$  is a ball of radius  $r > 0$  centered at  $y_x$ .

*Proof.* Consider

$$\left| \widetilde{F}_x(y) - F_x(y) \right| \leq \left| \widetilde{F}_x(y) - F_{x,N}(y) \right| + |F_{x,N}(y) - F_x(y)|.$$

We begin evaluating the second term on the right side of the above inequality. From  $\Gamma_x = F_x - \Phi_{d,x}$ , and applying Proposition 7 and eq. (2.40), we have:

$$(2.69) \quad \begin{aligned} |F_{x,N}(y) - F_x(y)| &= \left| \sum_{|\omega_y| \leq N} c_{\omega_y}(\Gamma_x) e^{i\omega_y y} - \Gamma_x(y) \right| \\ &= \left| \sum_{|\omega_y| > N} c_{\omega_y}(\Gamma_x) e^{i\omega_y y} \right| \leq C^{**} R_{\Gamma_{\mathbb{T}}} N^{-d-1}, \end{aligned}$$

where  $C^{**} \in \mathbb{R}^+$ .

Now we consider the first term. Letting  $\Theta_x := \widetilde{\Phi_{d,x}} - \Phi_{d,x}$  we claim that:

$$\begin{aligned} \left| \widetilde{F}_x(y) - F_{x,N}(y) \right| &= \left| \sum_{|\omega_y| \leq N} (c_{\omega_y}(\Phi_{d,x}) - c_{\omega_y}(\widetilde{\Phi_{d,x}})) e^{i\omega_y y} + \widetilde{\Phi_{d,x}}(y) - \Phi_{d,x}(y) \right| \\ &= \left| \sum_{|\omega_y| \leq N} c_{\omega_y}(\Theta_x) e^{i\omega_y y} - \Theta_x(y) \right|. \end{aligned}$$

Now using eqs. (2.61) and (2.67) we claim that there exist  $\beta_x(N)$  and  $\beta_{l,x}(N)$  for each  $0 \leq l \leq d$  s.t.:

$$(2.70) \quad \begin{aligned} \widetilde{y}_x &= y_x + \beta_x(N), & |\beta_x(N)| &\leq C_{2,d} \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{-d-2}, \\ \widetilde{A}_l(x) &= A_l(x) + \beta_{l,x}(N), & |\beta_{l,x}(N)| &\leq C_{6,d-l} \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{l-d-1}. \end{aligned}$$

Applying eq. (2.38) we get:

$$\begin{aligned}
\Theta_x(y) &= \sum_{l=0}^d \left( \widetilde{A}_l(x) V_l(y; \widetilde{y}_x) - A_l(x) V_l(y; y_x) \right) \\
&= \sum_{l=0}^d \left( \widetilde{A}_l(x) V_l(y; \widetilde{y}_x) - \left( \widetilde{A}_l(x) - \beta_l(N) \right) V_l(y; y_x) \right) \\
&= \sum_{l=0}^d \widetilde{A}_l(x) \left( V_l(y; \widetilde{y}_x) - V_l(y; y_x) \right) + \sum_{l=0}^d \beta_{l,x}(N) V_l(y; y_x) \\
&= \sum_{l=0}^d \widetilde{A}_l(x) \left( V_l(y; y_x) + \beta_{l,x}(N) - V_l(y; y_x) \right) + \sum_{l=0}^d \beta_{l,x}(N) V_l(y; y_x).
\end{aligned}$$

Now let  $r > 0$  be given. Since  $\inf_{x \in \mathbb{T}} |A_0(x)| = A_L > 0$ , for large enough  $N'$  we will have

$$(2.71) \quad \beta_x(N) < r \text{ for all } N > N', \text{ uniformly in } x.$$

For every  $\epsilon < r$ , we define:

$$U_{l,\epsilon}(y) := V_l(y; y_x + \epsilon) - V_l(y; y_x)$$

and take  $\epsilon = \beta_x(N)$  to get:

$$\Theta_x(y) = \sum_{l=0}^d \beta_{l,x}(N) V_l(y; y_x) + \sum_{l=0}^d \widetilde{A}_l(x) U_{l,\beta_x(N)}(y).$$

Once again we denote:

$$\begin{aligned}
\mathcal{Z}_x(y) &= \sum_{l=0}^d \beta_{l,x}(N) V_l(y; y_x) \\
\mathcal{W}_x(y) &= \sum_{l=0}^d \widetilde{A}_l(x) U_{l,\beta_x(N)}(y)
\end{aligned}$$

and write:

$$\Theta_x(y) = \mathcal{Z}_x(y) + \mathcal{W}_x(y).$$

Therefore

$$\begin{aligned}
\left| \widetilde{F}_x(y) - F_{x,N}(y) \right| &= \left| \sum_{|\omega_y| \leq N} c_{\omega_y}(\Theta_x) e^{i\omega_y y} - \Theta_x(y) \right| \\
&= \left| \sum_{|\omega_y| \leq N} \left( c_{\omega_y}(\mathcal{Z}_x) + c_{\omega_y}(\mathcal{W}_x) \right) e^{i\omega_y y} - \Theta_x(y) \right| \\
&= \left| \sum_{|\omega_y| \leq N} \left( c_{\omega_y}(\mathcal{Z}_x) + c_{\omega_y}(\mathcal{W}_x) \right) e^{i\omega_y y} - \mathcal{Z}_x(y) - \mathcal{W}_x(y) \right| \\
&\leq \left| \sum_{|\omega_y| \leq N} c_{\omega_y}(\mathcal{Z}_x) e^{i\omega_y y} - \mathcal{Z}_x(y) \right| + \left| \sum_{|\omega_y| \leq N} c_{\omega_y}(\mathcal{W}_x) e^{i\omega_y y} - \mathcal{W}_x(y) \right| \\
&= \left| \sum_{|\omega_y| > N} c_{\omega_y}(\mathcal{Z}_x) e^{i\omega_y y} \right| + \left| \sum_{|\omega_y| \leq N} c_{\omega_y}(\mathcal{W}_x) e^{i\omega_y y} - \mathcal{W}_x(y) \right|.
\end{aligned}$$

Considering  $\mathcal{Z}_x(y) = \sum_{l=0}^d \beta_{l,x}(N) V_l(y; y_x)$ , we have that:

$$c_{\omega_y}(\mathcal{Z}_x) = c_{\omega_y} \left( \sum_{l=0}^d \beta_{l,x}(N) V_l \right) = \sum_{l=0}^d \beta_{l,x}(N) c_{\omega_y}(V_l),$$

and since the function  $V_l$  is in  $C_{\mathbb{T}}^l$  there exists  $S_l \in \mathbb{R}^+$  such that:

$$\left| \sum_{|\omega_y| > N} c_{\omega_y}(V_l) e^{i\omega_y y} \right| \leq S_l \cdot N^{-l}$$

Consequently, we have that:

$$\begin{aligned} \left| \sum_{|\omega_y| > N} c_{\omega_y}(\mathcal{L}_x) e^{i\omega_y y} \right| &= \left| \sum_{|\omega_y| > N} \left( \sum_{l=0}^d \beta_{l,x}(N) c_{\omega_y}(V_l) \right) e^{i\omega_y y} \right| \\ &\leq \sum_{l=0}^d |\beta_{l,x}(N)| \left| \sum_{|\omega_y| > N} c_{\omega_y}(V_l) e^{i\omega_y y} \right| \\ &\leq (d+1) C_{6,d-l} \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{l-d-1} \sum_{|\omega_y| > N} |c_{\omega_y}(V_l)| \\ &\leq (d+1) C_{6,d-l} \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{l-d-1} S_l N^{-l} \\ &= (d+1) C_{6,d-l} S_l \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{-d-1}. \end{aligned}$$

Denoting:

$$(2.72) \quad C_{7,d} := (d+1) \cdot \max_{0 \leq l \leq d} \{C_{6,d-l} \cdot S_l\}$$

we get:

$$(2.73) \quad \left| \sum_{|\omega_y| > N} c_{\omega_y}(\mathcal{L}_x) e^{i\omega_y y} \right| \leq C_{7,d} \frac{A_x}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{-d-1}.$$

Now we move on to  $\mathcal{W}_x(y) = \sum_{l=0}^d \widetilde{A}_l(x) U_{l,\beta_x(N)}(y)$ . The function  $U_{l,\beta_x(N)}(y)$  is defined at the “*jump-free*” region so there exists  $C_4 \in \mathbb{R}$  (uniformly in  $x$ ) such that  $|U_{l,\beta_x(N)}(y)| \leq C_4 \epsilon$ . This bound can be obtained by Taylor-expanding the function  $V_l(y; y_x + \epsilon)$  at  $\epsilon = 0$ . In particular, for  $\epsilon = \beta_x(N)$  we have:

$$\begin{aligned} |\mathcal{W}_x(y)| &= \left| \sum_{l=0}^d \widetilde{A}_l(x) U_{l,\beta_x(N)}(y) \right| \leq \sum_{l=0}^d |\widetilde{A}_l(x)| |U_{l,\beta_x(N)}(y)| \leq \sum_{l=0}^d A_x \cdot C_4 \cdot |\beta_x(N)| \\ &\leq \sum_{l=0}^d \left( C_4 \cdot C_{2,d} \frac{A_x^2}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{-d-2} \right) \\ &= (d+1) \cdot C_4 \cdot C_{2,d} \frac{A_x^2}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{-d-2}. \end{aligned}$$

Denoting  $C_{8,d} = (d+1) \cdot C_{2,d} \cdot C_4$  we get:

$$(2.74) \quad |\mathcal{W}_x(y)| \leq C_{8,d} \frac{A_x^2}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{-d-2}.$$

Continuing our analysis we look at  $U_{l,\epsilon}(y)$ , for  $\epsilon = \beta_x(N)$ :

$$\begin{aligned} U_{l,\epsilon}(y) &= V_l(y; y_x + \epsilon) - V_l(y; y_x) \\ &= \frac{(2\pi)^l}{(l+1)!} \left( B_{l+1} \left( \frac{y - (y_x + \beta_x(N))}{2\pi} \right) - B_{l+1} \left( \frac{y - y_x}{2\pi} \right) \right) \\ &= \frac{(2\pi)^l}{(l+1)!} \left( B_{l+1} \left( \frac{y - \tilde{y}_x}{2\pi} \right) - B_{l+1} \left( \frac{y - y_x}{2\pi} \right) \right). \end{aligned}$$

Therefore, there exists a constant  $C_5$  such that in the region of length  $\beta_x(N)$  between  $y_x$  and  $\tilde{y}_x$  we have:

$$(2.75) \quad |U_{l,\beta_x(N)}(y)| \leq C_5 \cdot A_x.$$

As mentioned before, at the “*jump-free*” area we have:

$$(2.76) \quad |U_{l,\beta(N)}(y)| \leq C_4 C_{2,d} \frac{A_x^2}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{-d-2}.$$

We conclude that therefore exists a constant  $C_6$  such that the Fourier coefficients of  $\mathcal{W}_x$  are bounded by:

$$|c_{\omega_y}(\mathcal{W}_x)| \leq C_6 C_{2,d} \frac{A_x^2}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{-d-2}$$

which in turn allows us to conclude that exists another constant  $C_7$  such that:

$$(2.77) \quad \left| \sum_{|\omega_y| \leq N} c_{\omega_y}(\mathcal{W}_x) e^{i\omega_y y} \right| \leq C_7 C_{2,d} \frac{A_x^2}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{-d-1}.$$

Now by using eqs. (2.73) to (2.77) we get:

$$\begin{aligned} \left| \widetilde{F}_x(y) - F_{x,N}(y) \right| &\leq \left| \sum_{|\omega_y| > N} c_{\omega_y}(\mathcal{Z}_x) e^{i\omega_y y} \right| + \left| \sum_{|\omega_y| \leq N} c_{\omega_y}(\mathcal{W}_x) e^{i\omega_y y} - \mathcal{W}_x(y) \right| \\ &\leq \left| \sum_{|\omega_y| > N} c_{\omega_y}(\mathcal{Z}_x) e^{i\omega_y y} \right| + \left| \sum_{|\omega_y| \leq N} c_{\omega_y}(\mathcal{W}_x) e^{i\omega_y y} \right| + |\mathcal{W}_x(y)| \\ &\leq C_{7,d} \frac{A_x^2}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{-d-1} + C_7 C_{2,d} \frac{A_x^2}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{-d-1} + C_{8,d} \frac{A_x^2}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{-d-2} \\ &\leq (C_{7,d} + C_7 C_{2,d} + C_{8,d}) \frac{A_x^2}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{-d-1}. \end{aligned}$$

Denoting:

$$(2.78) \quad C_{9,d} := C_{7,d} + C_7 C_{2,d} + C_{8,d}$$

we have:

$$(2.79) \quad \left| \widetilde{F}_x(y) - F_{x,N}(y) \right| \leq C_{9,d} \frac{A_x^2}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{-d-1},$$

and by applying eqs. (2.69) and (2.79) we have:

$$(2.80) \quad \begin{aligned} \left| \widetilde{F}_x(y) - F_x(y) \right| &\leq \left| \widetilde{F}_x(y) - F_{x,N}(y) \right| + |F_{x,N}(y) - F_x(y)| \\ &\leq \left( C_{9,d} \frac{A_x^2}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} + C^{**} R_{\Gamma_\tau} \right) N^{-d-1}. \end{aligned}$$

By the definition of  $\mathcal{R}_{\Gamma|\gamma}$  (see eq. (2.58)) we claim that  $R_{\Gamma_\tau} \leq \mathcal{R}_{\Gamma|\gamma}$  which implies that

$$\left( C_{9,d} \frac{A_x^2}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} + C^{**} R_{\Gamma_\tau} \right) N^{-d-1} \leq \left( C_{9,d} \frac{A_x^2}{|A_0(x)|} + C^{**} \right) \mathcal{R}_{\Gamma|\gamma} N^{-d-1}.$$

Finally, by denoting

$$(2.81) \quad C_{10,d} = \max \{ C_{9,d}, C^{**} \}$$

we obtain

$$\left| \widetilde{F}_x(y) - F_x(y) \right| \leq C_{10,d} \frac{A_x(1 + A_x)}{|A_0(x)|} \mathcal{R}_{\Gamma|\gamma} N^{-d-1},$$

which completes the proof.  $\square$

**2.5. The full algorithm.** In this section we present the entire algorithm for reconstructing the slices  $F_x$  of  $F$  from its Fourier coefficients.

The 1D reconstruction procedure from [15, 10], is summarized in algorithm 2 below, while our 2D algorithm is presented in algorithm 3.

---

**Algorithm 1** Half-order reconstruction algorithm, [15]

---

Let  $f \in PC_{\mathbb{T}}^{(d+1,1)}$  and assume that  $f = \gamma + \varphi$  where  $\gamma \in C_{\mathbb{T}}^{(d+1)}$  and  $\phi$  is a piecewise polynomial of degree  $d$  absorbing all of  $f$ 's discontinuities.

**Require:** Fourier coefficients of  $\{c_k(f)\}_{|k| \leq M}$  s.t.  $M \gg 1$ .

- 1: Fix a reconstruction order  $d_1 \leq \lfloor \frac{d}{2} \rfloor$
- 2: **if** jump location unknown **then**
- 3:     Solve the polynomial in [15, eq. 3.3] with  $k = M$  and  $d = d_1$
- 4:     Take  $\tilde{\omega}$  to be the closest root to the unit circle
- 5:     Take  $\tilde{\xi} = \arg(\tilde{\omega})$  as the approximation of the actual discontinuity point
- 6: **else**
- 7:     Take  $\tilde{\xi} = x_0$
- 8: **end if**
- 9: Obtain the jump magnitudes  $\{\tilde{A}_l\}_{l=0}^{d_1}$  by using  $\tilde{\omega}$  and by solving the linear system in [15, eq. 3.5] with  $k = M, \dots, M + d_1$
- 10: Obtain coefficients of  $\tilde{\gamma}$ ,  $\{c_k(\tilde{\gamma})\}_{|k| \leq M}$  by [15, Algorithm, 2.2]
- 11: Take the final approximation

$$\tilde{f}(x) = \tilde{\gamma}(x) + \tilde{\phi}(x) = \sum_{|k| \leq M} c_k(\tilde{\gamma}) e^{ikx} + \sum_{l=0}^{d_1} \tilde{A}_l V_l(x; \tilde{\xi}).$$

---

**Algorithm 2** Full-order reconstruction algorithm, [10]

---

Let  $f : PC_{\mathbb{T}}^{(d+1,1)}$  and assume that  $f = \gamma + \varphi$  where  $\gamma \in C_{\mathbb{T}}^{(d+1)}$  and  $\phi$  is a piecewise polynomial of degree  $d$  absorbing all of  $f$ 's discontinuities.

**Require:** Fourier coefficients of  $\{c_k(f)\}_{|k| \leq M}$  s.t.  $M \gg 1$ .

- 1: Use algorithm 1 to approximate  $\tilde{\omega}_h = e^{-i\tilde{\xi}_h}$
- 2: Take  $N = \lfloor \frac{M}{d+2} \rfloor$
- 3: Construct the polynomial  $q_N^d(u) = \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} \tilde{m}_{(j+1)N} u^{d+1-j}$  where  $\tilde{m}_k := 2\pi(ik)^{d+1} c_k(f)$ , and find its roots.
- 4: Take  $\tilde{z}$  to be the closest root to the unit circle and denote  $\sqrt[N]{Z_{q_N^d}}$  as the set of  $N$  possible values of  $\sqrt[N]{\tilde{z}}$
- 5: Take  $\tilde{\omega}_f := \arg \min_{z \in \sqrt[N]{Z_{q_N^d}}} \{|z - \tilde{\omega}_h|\}$  and set  $\tilde{\xi}_f = -\arg(\tilde{\omega}_f)$
- 6: Recover the jump magnitudes by solving the perturbed linear system

$$\begin{bmatrix} \tilde{m}_N \tilde{\omega}_f^{-N} \\ \tilde{m}_{2N} \tilde{\omega}_f^{-2N} \\ \vdots \\ \tilde{m}_{(d+1)N} \tilde{\omega}_f^{-(d+1)N} \end{bmatrix} = V_N^d \cdot \begin{bmatrix} \tilde{\alpha}_0 \\ \tilde{\alpha}_1 \\ \vdots \\ \tilde{\alpha}_d \end{bmatrix}$$

and extracting  $\{\tilde{A}_l\}_{l=0}^d$  by using  $\tilde{A}_l = (-i)^{d-l} \tilde{\alpha}_{d-l}$

- 7: Obtain coefficients of  $\tilde{\gamma}$  using  $c_k(\tilde{\gamma}) = c_k(f) - c_k(\tilde{\phi})$ , where

$$c_k(\tilde{\phi}) = \frac{\tilde{\omega}_f^k}{2\pi} \sum_{l=0}^d \frac{\tilde{A}_l}{(ik)^{l+1}}$$

- 8: Take the final approximation

$$\tilde{f}(x) = \tilde{\gamma}(x) + \tilde{\phi}(x) = \sum_{|k| \leq M} c_k(\tilde{\gamma}) e^{ikx} + \sum_{l=0}^{d_1} \tilde{A}_l V_l(x; \tilde{\xi}).$$

---

**Algorithm 3** 2D reconstruction
 

---

Let  $F : \mathbb{T}^2 \rightarrow \mathbb{R}$  and  $F_x : \mathbb{T} \rightarrow \mathbb{R}$  as defined in definition 3 and  $\psi_{\omega_y} \in PC_{\mathbb{T}}^{(d+1,1)}$  as defined in eq. (1.6) and Lemma 1.1.

**Require:**  $(2M + 1) \times (2N + 1)$  Fourier coefficients of  $F$  s.t.  $N^2 \leq M$  and a sets of samples  $X = \{x_0, \dots, x_n\} \subset \mathbb{T}$ ,  $Y = \{y_0, \dots, y_m\} \subset \mathbb{T}$

- 1: **for**  $x \in X$  **do**
- 2:     **for**  $\omega_y \leftarrow -N \rightarrow N$  **do**
- 3:          $\widetilde{\psi}_{\omega_y}(x) \leftarrow$  apply algorithm 2 using  $\{c_{\omega_x}(\psi_{\omega_y})\}_{|\omega_x| \leq M}$  and known jump location  $x_0 = -\pi$
- 4:     **end for**
- 5:     **for**  $y \in Y$  **do**
- 6:          $\widetilde{F}_x(y) \leftarrow$  apply algorithm 2 using  $\{\widetilde{\psi}_{\omega_y}(x)\}_{|\omega_y| \leq N}$
- 7:     **end for**
- 8: **end for**

---

### 3. NUMERICAL EXPERIMENTS

In this section we provide simulations with the primary goal of validating the asymptotic accuracy predictions developed in this work. We provide the actual Fourier coefficients, which were calculated analytically, for the function  $F : \mathbb{T}^2 \rightarrow \mathbb{R}$  which is defined as follows:

$$(3.1) \quad U_n(y) := \begin{cases} B_{n+1}(\frac{y+2\pi}{2\pi}), & y \in [-2\pi, 0) \\ B_n(\frac{y}{2\pi}), & y \in [0, 2\pi) \end{cases}$$

as  $B_n(y)$  is the Bernoulli function [64]

$$V_n(x, y) := -\frac{(2\pi)^n}{(n+1)!} U_n(y - \xi(x))$$

$$\Phi_d(x, y) := \sum_{l=0}^d A_l V_l(x, y)$$

where  $\xi(x)$  is the discontinuity curve and  $A_l(x)$  is the jump magnitude at  $x$ . Setting  $\xi(x) = x$  and  $A_l(x) = 1$  as  $l = 0, \dots, d$  we simulate  $F : \mathbb{T}^2 \rightarrow \mathbb{R}$  which is defined by

$$(3.2) \quad F(x, y) := \Phi_{11}(x, y)$$

In our simulation we fix a point  $x \in \mathbb{T}$  to get the slice  $F_x$ . We set the reconstruction orders of  $\psi_{\omega_y}$  and of  $F_x$  to 9, i.e.  $d = 9$ . We further set number of the given Fourier coefficients for the unknown function  $F : \mathbb{T}^2 \rightarrow \mathbb{R}$  as different sets of  $(M, N)$  where  $(M, N) = (N^2, N)$ . In the figures below we calculate the following approximation errors:

- (1) The error in approximating  $F_x$  over a set  $Y \subset \mathbb{T}$

$$\Delta F_x := \max \left\{ \left| F_x(y) - \widetilde{F}_x(y) \right| : y \in \mathbb{T} \right\}, \quad x \in \mathbb{T}$$

which is shown in fig. 3a.

- (2) The error in approximating the discontinuity curve  $\xi(x)$

$$\Delta \xi := \max_{x \in \mathbb{T}} \left\{ \left| \xi(x) - \widetilde{\xi}(x) \right| \right\}$$

which is shown in fig. 3b.

- (3) The error in approximating the jump magnitudes:

$$\Delta A_l := \max_{x \in \mathbb{T}} \left\{ \left| A_l(x) - \widetilde{A}_l(x) \right| \right\}, \quad l = 0, \dots, d$$

which is shown in fig. 3c.

The simulations were performed in a standard Python environment using the mpmath library which in turn allowed us to perform our calculations with a varying precision up to 100 decimal places.

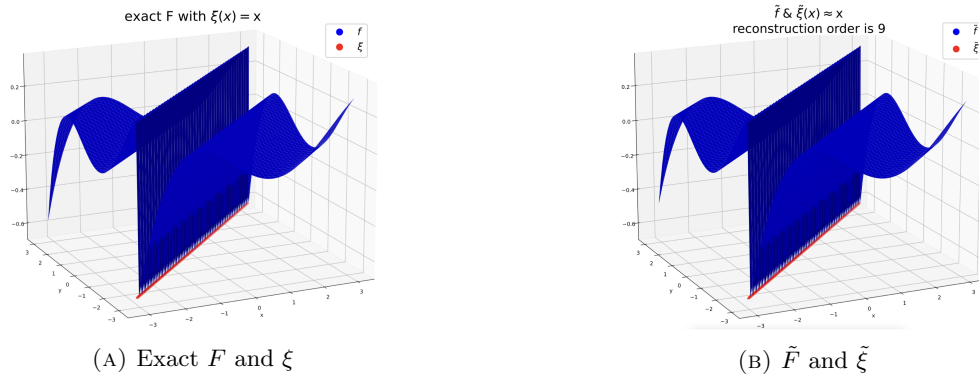


FIGURE 2. Exact function with its discontinuity curve and approximated functions with its approximated discontinuity curve

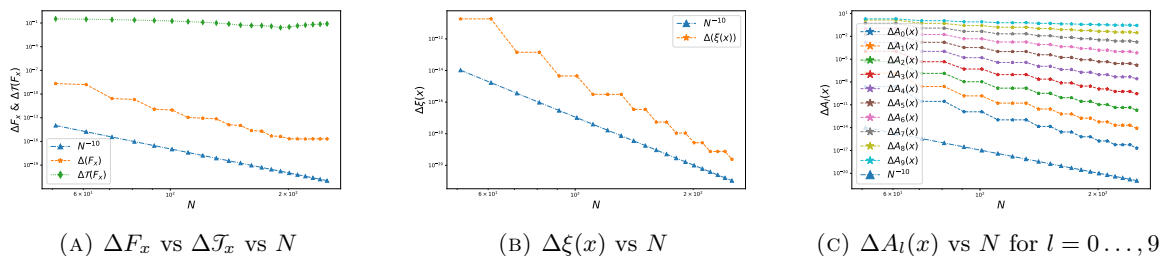


FIGURE 3. Approximation errors computing by our algorithm for the function in fig. 2b.

In fig. 3a we present the maximal error in approximating  $F_x$  at  $x = 1.1$  using our method vs the maximal error in approximating  $F_x$  using a truncated Fourier sum. In fig. 3b we present the maximal error in approximating the jump location of  $F_x$  at  $x = 1.1$  ( $\tilde{\xi}(1.1)$ ) using our method and in fig. 3c we present maximal error in approximating  $\frac{d^i}{dx^i}(F_x)$  as  $0 \leq i \leq 9$  jump magnitudes at  $x = 1.1$ . All plots include a graph of  $N^{-10}$  for comparison with the theoretical bounds presented in theorem 1.

The code implementing the algorithms described in section 2.5 and reproducing the results of this section is available at <https://github.com/mlevinov/algebraic-fourier-2d>.

#### 4. DISCUSSION AND FUTURE WORK

In this paper we proposed a method for reconstructing a 2D function  $F : \mathbb{T}^2 \rightarrow \mathbb{R}$  from its Fourier coefficients. Our model of the function was restricted to only two continuity pieces. We have demonstrated algebraic convergence of the method, with rates commensurate with the smoothness of the pieces. There exist several natural extensions of our work in multiple directions.

- (1) In order to recover the entire function, one would need to perform interpolation from the approximated slices, while taking into account the discontinuity curve as well.
- (2) Our method can be extended to reconstructing functions with multiple discontinuity pieces, supported on domains with smooth boundaries  $\{\Sigma_1, \dots, \Sigma_K\}$ . First, the discontinuity structure can be approximated by a low-order method (as done in the one-dimensional case in [15, 10]). Then, each piece can be further localized by applying a mollifier to create a region containing a single discontinuity curve. An advantage of this approach is that the boundary discontinuity will disappear, enabling to apply fast algorithms in the first stage.
- (3) The algebraic structure of the method can be extended to handle other orthonormal basis, such as Chebyshev polynomials. Related work in this direction can be found in [31, 32]. This would allow us to handle non-periodic functions with variable discontinuities more efficiently.
- (4) The fundamental relationship between  $\hat{F}$  and  $\psi_{\omega,y}$  can be naturally extended to multiple dimensions by recursion. While this introduces additional symbolic complexity, we believe these challenges can be efficiently handled by modern computer algebra and automatic differentiation techniques.



## REFERENCES

- [1] Ben Adcock, Milana Gataric, and Anders C. Hansen. “Weighted Frames of Exponentials and Stable Recovery of Multidimensional Functions from Nonuniform Fourier Samples”. In: *Applied and Computational Harmonic Analysis* 42.3 (May 1, 2017), pp. 508–535. ISSN: 1063-5203. DOI: 10.1016/j.acha.2015.09.006.
- [2] Ben Adcock, Daan Huybrechs, and Jesús Martín-Vaquero. “On the Numerical Stability of Fourier Extensions”. In: *Foundations of Computational Mathematics* 14.4 (Aug. 2014), pp. 635–687. ISSN: 1615-3383. DOI: 10.1007/s10208-013-9158-8. (Visited on 05/16/2021).
- [3] Andrey Akinshin, Gil Goldman, and Yosef Yomdin. “Geometry of Error Amplification in Solving the Prony System with Near-Colliding Nodes”. In: *Mathematics of Computation* 90.327 (2021), pp. 267–302. ISSN: 0025-5718, 1088-6842. DOI: 10.1090/mcom/3571. (Visited on 12/20/2020).
- [4] Sergio Amat et al. “Global and Explicit Approximation of Piecewise-Smooth Two-Dimensional Functions from Cell-Average Data”. In: *IMA Journal of Numerical Analysis* 43.4 (2023), pp. 2299–2319. ISSN: 0272-4979. DOI: 10.1093/imanum/drac042.
- [5] Anat Amir and David Levin. “High Order Approximation to Non-Smooth Multivariate Functions”. In: *Computer Aided Geometric Design* 63 (2018), pp. 31–65. ISSN: 0167-8396. DOI: 10.1016/j.cagd.2018.02.004.
- [6] A. Barkhudaryan, R. Barkhudaryan, and A. Poghosyan. “Asymptotic Behavior of Eckhoff’s Method for Fourier Series Convergence Acceleration”. In: *Analysis in Theory and Applications* 23.3 (2007), pp. 228–242.
- [7] D. Batenkov, V. Golubyatnikov, and Y. Yomdin. “Reconstruction of Planar Domains from Partial Integral Measurements”. In: *Complex Analysis and Dynamical Systems V*. Vol. 591. Contemp. Math. Amer. Math. Soc., Providence, RI, 2013, pp. 51–66. (Visited on 12/23/2017).
- [8] D. Batenkov and Y. Yomdin. “On the Accuracy of Solving Confluent Prony Systems”. In: *SIAM J. Appl. Math.* 73.1 (2013), pp. 134–154. DOI: 10.1137/110836584.
- [9] Dmitry Batenkov. “Decimated Generalized Prony Systems”. In: *arXiv:1308.0753 [math]* (Aug. 2013). arXiv: 1308.0753 [math]. (Visited on 06/01/2014).
- [10] Dmitry Batenkov. “Complete algebraic reconstruction of piecewise-smooth functions from Fourier data”. In: *Mathematics of Computation* 84.295 (2015), pp. 2329–2350.
- [11] Dmitry Batenkov. “Accurate Solution of Near-Colliding Prony Systems via Decimation and Homotopy Continuation”. In: *Theoretical Computer Science*. Symbolic Numeric Computation 681 (June 2017), pp. 27–40. ISSN: 0304-3975. DOI: 10.1016/j.tcs.2017.03.026. (Visited on 06/13/2017).
- [12] Dmitry Batenkov. “Stability and Super-Resolution of Generalized Spike Recovery”. In: *Applied and Computational Harmonic Analysis* 45.2 (Sept. 2018), pp. 299–323. ISSN: 1063-5203. DOI: 10.1016/j.acha.2016.09.004. (Visited on 07/10/2018).
- [13] Dmitry Batenkov and Nuha Diab. “Super-Resolution of Generalized Spikes and Spectra of Confluent Vandermonde Matrices”. In: *Applied and Computational Harmonic Analysis* 65 (July 2023), pp. 181–208. ISSN: 1063-5203. DOI: 10.1016/j.acha.2023.03.002. (Visited on 03/20/2023).
- [14] Dmitry Batenkov, Gil Goldman, and Yosef Yomdin. “Super-Resolution of near-Colliding Point Sources”. In: *Information and Inference: A Journal of the IMA* 10.2 (June 2021), pp. 515–572. DOI: 10.1093/imaiai/iaaa005. (Visited on 06/29/2021).
- [15] Dmitry Batenkov and Yosef Yomdin. “Algebraic Fourier reconstruction of piecewise smooth functions”. In: *Mathematics of Computation* 81.277 (2012), pp. 277–318.
- [16] Dmitry Batenkov and Yosef Yomdin. “Geometry and Singularities of the Prony Mapping”. In: *Journal of Singularities* 10 (2014), pp. 1–25. ISSN: 19492006. DOI: 10.5427/jsing.2014.10a. (Visited on 12/25/2014).
- [17] Dmitry Batenkov et al. “Conditioning of Partial Nonuniform Fourier Matrices with Clustered Nodes”. In: *SIAM Journal on Matrix Analysis and Applications* 44.1 (Jan. 2020), pp. 199–220. ISSN: 0895-4798. DOI: 10/ggjwzb. (Visited on 02/02/2020).
- [18] Dmitry Batenkov et al. “The Spectral Properties of Vandermonde Matrices with Clustered Nodes”. In: *Linear Algebra and its Applications* 609 (Jan. 2021), pp. 37–72. ISSN: 0024-3795. DOI: 10.1016/j.laa.2020.08.034. (Visited on 09/07/2020).
- [19] Chris Blakely, Anne Gelb, and Antonio Navarra. “An automated method for recovering piecewise smooth functions on spheres free from Gibbs oscillations”. In: *Sampling Theory in Signal and Image Processing* 6.3 (Sept. 2007), pp. 323–346. ISSN: 1530-6429. URL: <https://asu.pure.elsevier.com/en/publications/an-automated-method-for-recovering-piecewise-smooth-functions-on-> (visited on 03/03/2017).

- [20] George Boole and John Fletcher Moulton. *A treatise on the calculus of finite differences*. Macmillan and company, 1872.
- [21] Emmanuel J Candès and David L Donoho. “New tight frames of curvelets and optimal representations of objects with piecewise C2 singularities”. In: *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences* 57.2 (2004), pp. 219–266.
- [22] Emmanuel J. Candès and Carlos Fernandez-Granda. “Towards a Mathematical Theory of Super-resolution”. In: *Communications on Pure and Applied Mathematics* 67.6 (June 2014), pp. 906–956. ISSN: 1097-0312. DOI: 10.1002/cpa.21455. (Visited on 05/18/2014).
- [23] Zheng Chen and Chi-Wang Shu. “Recovering Exponential Accuracy from Collocation Point Values of Smooth Functions with End-Point Singularities”. In: *Journal of Computational and Applied Mathematics* 265 (Aug. 2014), pp. 83–95. ISSN: 03770427. DOI: 10.1016/j.cam.2013.09.029. (Visited on 12/15/2024).
- [24] Albert Cohen, Olga Mula, and Agustín Somacal. *High Order Recovery of Geometric Interfaces from Cell-Average Data*. Feb. 1, 2024. DOI: 10.48550/arXiv.2402.00946. arXiv: 2402.00946 [cs, math]. preprint.
- [25] Annie Cuyt and Wen-shin Lee. “Multivariate Exponential Analysis from the Minimal Number of Samples”. In: *Advances in Computational Mathematics* 44.4 (Aug. 2018), pp. 987–1002. ISSN: 1572-9044. DOI: 10.1007/s10444-017-9570-8. (Visited on 07/04/2019).
- [26] I. Daubechies. *Ten Lectures on Wavelets*. CBMS-NSF Regional Conference Series in Applied Mathematics. Society for Industrial and Applied Mathematics, Jan. 1, 1992. 369 pp. ISBN: 978-0-89871-274-2. URL: <http://epubs.siam.org/doi/book/10.1137/1.9781611970104> (visited on 02/01/2016).
- [27] Nuha Diab and Dmitry Batenkov. *Spectral Properties of Infinitely Smooth Kernel Matrices in the Single Cluster Limit, with Applications to Multivariate Super-Resolution*. July 2024. arXiv: 2407.10600 [cs, math]. (Visited on 07/16/2024).
- [28] Benedikt Diederichs, Mihail N. Kolountzakis, and Effie Papageorgiou. “How Many Fourier Coefficients Are Needed?” In: *Monatshefte für Mathematik* (Oct. 2022). ISSN: 0026-9255, 1436-5081. DOI: 10.1007/s00605-022-01792-0. (Visited on 11/08/2022).
- [29] David L. Donoho. “Wedgelets: Nearly Minimax Estimation of Edges”. In: *The Annals of Statistics* 27.3 (June 1999), pp. 859–897. ISSN: 0090-5364, 2168-8966. DOI: 10.1214/aos/1018031261.
- [30] Knut Eckhoff. “On a high order numerical method for functions with singularities”. In: *Mathematics of Computation* 67.223 (1998), pp. 1063–1087.
- [31] Knut S Eckhoff. “Accurate and efficient reconstruction of discontinuous functions from truncated series expansions”. In: *mathematics of computation* 61.204 (1993), pp. 745–763.
- [32] Knut S Eckhoff. “Accurate reconstructions of functions of finite regularity from truncated Fourier series expansions”. In: *Mathematics of Computation* 64.210 (1995), pp. 671–690.
- [33] F. Filbir, H. N. Mhaskar, and J. Prestin. “On the Problem of Parameter Estimation in Exponential Sums”. In: *Constructive Approximation* 35.3 (2012), pp. 323–343. (Visited on 05/10/2015).
- [34] Guido Fubini. “Sugli integrali multipli”. In: *Rend. Acc. Naz. Lincei* 16 (1907), pp. 608–614.
- [35] Anne Gelb and Sigal Gottlieb. “The Resolution of the Gibbs Phenomenon for Fourier Spectral Methods”. In: *Advances in The Gibbs Phenomenon. Sampling Publishing, Potsdam, New York* (2007). (Visited on 02/26/2014).
- [36] Anne Gelb and Guohui Song. “Detecting Edges from Non-uniform Fourier Data Using Fourier Frames”. In: *Journal of Scientific Computing* (Nov. 18, 2016), pp. 1–22. ISSN: 0885-7474, 1573-7691. DOI: 10.1007/s10915-016-0320-8.
- [37] J Willard Gibbs. “Fourier’s series”. In: *Nature* 59.1522 (1898), pp. 200–200.
- [38] D. Gottlieb and C. W. Shu. “On the Gibbs Phenomenon III: Recovering Exponential Accuracy in a Sub-Interval from a Spectral Partial Sum of a Piecewise Analytic Function”. In: *SIAM Journal on Numerical Analysis* (1996), pp. 280–290.
- [39] D. Gottlieb and C. W. Shu. “On the Gibbs Phenomenon and Its Resolution”. In: *SIAM Review* (1997), pp. 644–668.
- [40] David Gottlieb and Steven A Orszag. *Numerical Analysis of Spectral Methods: Theory and Applications*. Vol. 26. SIAM, 1977.
- [41] David Gottlieb and Steven A Orszag. *Numerical analysis of spectral methods: theory and applications*. SIAM, 1977.

- [42] Kanghui Guo and Demetrio Labate. “Optimally Sparse Representations of 3D Data with  $C^2$  Surface Singularities Using Parseval Frames of Shearlets”. In: *SIAM Journal on Mathematical Analysis* 44.2 (2012), pp. 851–886.
- [43] Rami Katz, Nuha Diab, and Dmitry Batenkov. “Decimated Prony’s Method for Stable Super-Resolution”. In: *IEEE Signal Processing Letters* 30 (2023), pp. 1467–1471. ISSN: 1558-2361. DOI: 10.1109/LSP.2023.3324553. (Visited on 10/25/2023).
- [44] Rami Katz, Nuha Diab, and Dmitry Batenkov. “On the Accuracy of Prony’s Method for Recovery of Exponential Sums with Closely Spaced Exponents”. In: *Applied and Computational Harmonic Analysis* 73 (Nov. 2024), p. 101687. ISSN: 1063-5203. DOI: 10.1016/j.acha.2024.101687. (Visited on 07/17/2024).
- [45] Rami Katz, Giulia Giordano, and Dmitry Batenkov. “Data-Driven Delay Estimation in Reaction-Diffusion Systems via Exponential Fitting”. In: *IFAC-PapersOnLine*. 18th IFAC Workshop on Time Delay Systems TDS 2024 58.27 (Jan. 2024), pp. 102–107. ISSN: 2405-8963. DOI: 10.1016/j.ifacol.2024.10.307. (Visited on 11/28/2024).
- [46] Stefan Kunis et al. “A Multivariate Generalization of Prony’s Method”. In: *Linear Algebra and its Applications* 490 (Feb. 2016), pp. 31–47. ISSN: 0024-3795. DOI: 10/f797mv. (Visited on 10/28/2019).
- [47] Gitta Kutyniok and Demetrio Labate, eds. *Shearlets*. Boston: Birkhäuser Boston, 2012. ISBN: 978-0-8176-8315-3 978-0-8176-8316-0. DOI: 10.1007/978-0-8176-8316-0.
- [48] G. Kvernadze. “Approximating the Jump Discontinuities of a Function by Its Fourier-Jacobi Coefficients”. In: *Mathematics of Computation* 73.246 (2004), pp. 731–752.
- [49] G. Kvernadze. “Approximation of the Discontinuities of a Function by Its Classical Orthogonal Polynomial Fourier Coefficients”. In: *Mathematics of Computation* 79 (2010), pp. 2265–2285.
- [50] David Levin. “Reconstruction of Piecewise Smooth Multivariate Functions from Fourier Data”. In: *Axioms* 9.3 (July 24, 2020), p. 88. ISSN: 20751680. DOI: 10.3390/axioms9030088.
- [51] Yaron Lipman and David Levin. “Approximating Piecewise-Smooth Functions”. In: *IMA Journal of Numerical Analysis* 30.4 (2010), pp. 1159–1183.
- [52] Stéphane Mallat. *A wavelet tour of signal processing*. Elsevier, 1999.
- [53] H. N. Mhaskar. “Super-Resolution Meets Machine Learning: Approximation of Measures”. In: *Journal of Fourier Analysis and Applications* (2019), pp. 1–19.
- [54] Hrushikesh N Mhaskar, Paul Nevai, and Eugene Shvarts. “Applications of classical approximation theory to periodic basis function networks and computational harmonic analysis”. In: *Bulletin of Mathematical Sciences* 3.3 (2013), pp. 485–549.
- [55] Hrushikesh Narhar Mhaskar and Jürgen Prestin. “On the Detection of Singularities of a Periodic Function”. In: *Advances in Computational Mathematics* 12.2-3 (2000), pp. 95–131. (Visited on 05/24/2015).
- [56] Frank WJ Olver et al. *NIST handbook of mathematical functions hardback and CD-ROM*. Cambridge university press, 2010.
- [57] Peter J Olver. *Applications of Lie groups to differential equations*. Vol. 107. Springer Science & Business Media, 1993.
- [58] Arnak Poghosyan, Lusine Poghosyan, and Rafayel Barkhudaryan. “On the Convergence of the Quasi-Periodic Approximations on a Finite Interval”. In: *Armenian Journal of Mathematics* 13.10 (Dec. 2021), pp. 1–44. ISSN: 1829-1163. DOI: 10.52737/18291163-2021.13.10-1-44. (Visited on 08/11/2024).
- [59] Clarice Poon and Gabriel Peyré. “MultiDimensional Sparse Super-Resolution”. In: *SIAM Journal on Mathematical Analysis* 51.1 (Jan. 2019), pp. 1–44. ISSN: 0036-1410, 1095-7154. DOI: 10.1137/17M1147822. (Visited on 04/16/2019).
- [60] Tomas Sauer. “Prony’s Method in Several Variables”. In: *Numerische Mathematik* 136.2 (June 2017), pp. 411–438. ISSN: 0029-599X, 0945-3245. DOI: 10.1007/s00211-016-0844-8. (Visited on 03/04/2019).
- [61] Tomas Sauer. “Prony’s Method in Several Variables: Symbolic Solutions by Universal Interpolation”. In: *Journal of Symbolic Computation* 84 (Jan. 2018), pp. 95–112. ISSN: 07477171. DOI: 10.1016/j.jsc.2017.03.006. (Visited on 03/04/2019).
- [62] Kevin Schober, Jürgen Prestin, and Serhii A. Stasyuk. “Edge Detection with Trigonometric Polynomial Shearlets”. In: *Advances in Computational Mathematics* 47.1 (Feb. 2021), p. 17. ISSN: 1019-7168, 1572-9044. DOI: 10.1007/s10444-020-09838-3. (Visited on 11/21/2022).
- [63] James Stewart. *Calculus: early transcendentals 8th edition*. Cengage Learning, 2016.

- [64] Zhi-Wei Sun and Hao Pan. “Identities concerning Bernoulli and Euler polynomials”. In: *arXiv preprint math/0409035* (2004).
- [65] E. Tadmor. “Filters, Mollifiers and the Computation of the Gibbs Phenomenon”. In: *Acta Numerica* 16 (2007), pp. 305–378.
- [66] Jelle Veraart et al. “Gibbs ringing in diffusion MRI”. In: *Magnetic resonance in medicine* 76.1 (2016), pp. 301–314.
- [67] Gabriel Wasserman, Rick Archibald, and Anne Gelb. “Image Reconstruction from Fourier Data Using Sparsity of Edges”. In: *Journal of Scientific Computing* 65.2 (Nov. 1, 2015), pp. 533–552. ISSN: 0885-7474, 1573-7691. DOI: 10.1007/s10915-014-9973-3.
- [68] Antoni Zygmund. *Trigonometric series*. Vol. 1. Cambridge university press, 2002.

## APPENDIX A. APPENDIX

**A.1. Auxiliary results.** We first provide several auxiliary results.

**Lemma A.1.** *Let  $\omega_y \in \mathbb{Z}$ , and  $x \in \mathbb{T}$ ,  $\xi(x)$  as defined in definition 3 and  $A_m(x)$  as defined in eq. (1.5). Then by using the following notations*

- $\xi_i(x) := \frac{d^i}{dx^i} \xi(x)$
- $f_c(x) := \cos(\omega_y \xi(x))$
- $f_s(x) := \sin(\omega_y \xi(x))$
- $g_c(x) := f_c(x) \cdot \xi_1(x)$
- $g_s(x) := f_s(x) \cdot \xi_1(x)$

we have:

$$\begin{aligned} \frac{d^i}{dx^i} g_c(x) &= \frac{d^i}{dx^i} (f_c(x) \cdot \xi_1(x)) = \sum_{j=0}^i a_j \cdot f_c^{(j)}(x) \cdot \xi_{i+1-j}(x) \\ \frac{d^i}{dx^i} g_s(x) &= \frac{d^i}{dx^i} (f_s(x) \cdot \xi_1(x)) = \sum_{j=0}^i b_j \cdot f_s^{(j)}(x) \cdot \xi_{i+1-j}(x) \end{aligned}$$

where  $a_j, b_j \in \mathbb{R}$  for each  $1 \leq j \leq i$ .

*Proof.* We will provide a proof only for  $f_c(x)$ , the proof for  $f_s(x)$  is identical. First, by using the general Leibnitz rule [63, Section 3.3] we have

$$\begin{aligned} \frac{d^l}{dx^l} (g_c(x) \cdot A_m(x)) &= \sum_{i=0}^l a_i \cdot g_c^{(i)}(x) \cdot A_m^{(l-i)}(x) \\ \frac{d^l}{dx^l} (g_s(x) \cdot A_m(x)) &= \sum_{i=0}^l a_i \cdot g_s^{(i)}(x) \cdot A_m^{(l-i)}(x) \end{aligned} \tag{A.1}$$

where  $a_i \in \mathbb{R}$  for each  $1 \leq i \leq l$ . Now, for  $i = 1$  we have

$$g_c^{(1)}(x) = (f_c(x) \cdot \xi_1(x))^{(1)} = f_c^{(1)}(x) \cdot f_c(x) \cdot \xi_2(x) = \sum_{j=0}^1 f_c^{(j)}(x) \cdot \xi_{2-j}(x).$$

Assuming that

$$\frac{d^i}{dx^i} g_c(x) = \frac{d^i}{dx^i} (f_c(x) \cdot \xi_1(x)) = \sum_{j=0}^i a_j \cdot f_c^{(j)}(x) \cdot \xi_{i+1-j}(x),$$

we continue to

$$\begin{aligned} \frac{d^{i+1}}{dx^{i+1}} g_c(x) &= \frac{d}{dx} \left( \frac{d^i}{dx^i} g_c(x) \right) = \frac{d}{dx} \left( \sum_{j=0}^i a_j \cdot f_c^{(j)}(x) \cdot \xi_{i+1-j}(x) \right) = \\ &= \sum_{j=0}^i a_j \cdot \frac{d}{dx} \left( f_c^{(j)}(x) \cdot \xi_{i+1-j}(x) \right) = \sum_{j=0}^i a_j \cdot f_c^{(j+1)}(x) \cdot \xi_{i+1-j}(x) + \sum_{j=0}^i a_j \cdot f_c^{(j)}(x) \cdot \xi_{i+2-j}(x). \end{aligned}$$

For the first sum, we denote  $k = j + 1$  and get

$$\sum_{j=0}^i a_j \cdot f_c^{(j+1)}(x) \cdot \xi_{i+1-j}(x) = \sum_{k=1}^{i+1} a_{k-1} \cdot f_c^{(k)}(x) \cdot \xi_{i+2-k}(x),$$

so reverting back to  $j$  we get

$$\begin{aligned}
& \sum_{j=0}^i f_c^{(j+1)}(x) \cdot \xi_{i+1-j}(x) + \sum_{j=0}^i f_c^{(j)}(x) \cdot \xi_{i+2-j}(x) = \\
& \sum_{j=1}^{i+1} a_{j-1} \cdot f_c^{(j)}(x) \cdot \xi_{i+2-j}(x) + \sum_{j=0}^i a_j \cdot f_c^{(j)}(x) \cdot \xi_{i+2-j}(x) = \\
& a_0 \cdot f_c^{(0)}(x) \cdot \xi_{i+2}(x) + \sum_{j=1}^i a_j \cdot f_c^{(j)}(x) \cdot \xi_{i+2-j}(x) + \sum_{j=1}^i a_{j-1} \cdot f_c^{(j)}(x) \cdot \xi_{i+2-j}(x) + a_i \cdot f_c^{(i+1)}(x) \cdot \xi_1(x) \\
& = a_0 \cdot f_c^{(0)}(x) \cdot \xi_{i+2}(x) + \sum_{j=1}^i (a_{j-1} + a_j) \cdot f_c^{(j)}(x) \cdot \xi_{i+2-j}(x) + a_i \cdot f_c^{(i+1)}(x) \cdot \xi_1(x).
\end{aligned}$$

Denoting  $c_j = a_{j-1} + a_j$  we get  $c_j \in \mathbb{R}$  and

$$\frac{d^{i+1}}{dx^{i+1}} g_c(x) = \sum_{j=0}^{i+1} c_j \cdot f_c^{(j)}(x) \cdot \xi_{i+2-j}(x). \quad \square$$

**Lemma A.2.** *Let  $x \in \mathbb{T}$ ,  $\omega_y \in \mathbb{Z}$  and let  $\xi_i(x)$ ,  $f_c(x)$ ,  $f_s(x)$  be as defined in Lemma A.1. Furthermore:*

- $\omega_{\pm, y}^a := -\omega_y^a$  or  $\omega_y^a$ , where  $a \in \mathbb{N}$ .
- $f_{c|s}^{(k)}(x) := \frac{d^k}{dx^k} \cos(\omega_y \xi(x))$  or  $\frac{d^k}{dx^k} \sin(\omega_y \xi(x))$ .
- $g_{k, m_l, a_l}(x) = \prod_{j=1}^k a_{l, j} \cdot \xi_j^{m_{l, j}}$ , where  $a_{l, j} \in \mathbb{R}$  and  $m_{l, j} \in \mathbb{N}$  as  $1 \leq j \leq k$ .

Then there exists  $1 \leq s \in \mathbb{N}$  s.t.

$$\frac{d^k}{dx^k} f_{c|s}(x) = \sum_{l=1}^s \omega_{\pm, y}^{m_{l, 0}} \cdot f_{c|s}(x) \cdot g_{k, m_l, a_l}(x)$$

where  $m_{1, 0} = \max_{1 \leq l \leq s} \{m_{l, 0}\} = k$  and  $m_{1, 1} = \max_{1 \leq l \leq s} \{m_{l, j} \mid 1 \leq j \leq k\} = k$ .

*Proof.* First we see that

$$(A.2) \quad \frac{d}{dx} f_{c|s}(x) = \omega_{\pm, y} \cdot f_{c|s}(x) \cdot \xi_1$$

Now we assume that for  $k \in \mathbb{N}$  s.t.  $k \leq d$  we have

$$(A.3) \quad \frac{d^k}{dx^k} f_{c|s}(x) = \sum_{l=1}^s \omega_{\pm, y}^{m_{l, 0}} \cdot f_{c|s}(x) \cdot g_{k, m_l, a_l}(x),$$

where  $m_{1, 0} = \max_{1 \leq l \leq s} \{m_{l, 0}\} = k$  and  $m_{1, 1} = \max_{1 \leq l \leq s} \{m_{l, j} \mid 1 \leq j \leq k\} = k$ . Presently we show that there

exists  $s_1 \in \mathbb{N}$  s.t.  $\frac{d^{k+1}}{dx^{k+1}} f_{c|s}(x) = \sum_{l=1}^{s_1} \omega_{\pm, y}^{m_{l, 0}} \cdot f_{c|s}(x) \cdot g_{k, m_l, a_l}(x)$ :

$$\begin{aligned}
\frac{d^{k+1}}{dx^{k+1}} f_{c|s}(x) &= \frac{d}{dx} \left( \frac{d^k}{dx^k} f_{c|s}(x) \right) \stackrel{\text{eq. (A.3)}}{=} \frac{d}{dx} \left( \sum_{l=1}^s \omega_{\pm, y}^{m_{l, 0}} \cdot f_{c|s}(x) \cdot g_{k, m_l, a_l}(x) \right) \\
&= \sum_{l=1}^s \omega_{\pm, y}^{m_{l, 0}} \cdot \left( \frac{d}{dx} (f_{c|s}(x)) \cdot g_{k, m_l, a_l}(x) + f_{c|s}(x) \cdot \frac{d}{dx} (g_{k, m_l, a_l}(x)) \right) \\
&= \sum_{l=1}^s \omega_{\pm, y}^{m_{l, 0}} \cdot \frac{d}{dx} (f_{c|s}(x)) \cdot g_{k, m_l, a_l}(x) + \sum_{l=1}^s \omega_{\pm, y}^{m_{l, 0}} \cdot f_{c|s}(x) \cdot \frac{d}{dx} (g_{k, m_l, a_l}(x)).
\end{aligned}$$

Now, by denoting

$$(A.4) \quad \Sigma_1(x) := \sum_{l=1}^s \omega_{\pm, y}^{m_{l, 0}} \cdot \frac{d}{dx} (f_{c|s}(x)) \cdot g_{k, m_l, a_l}(x)$$

$$(A.5) \quad \Sigma_2(x) := \sum_{l=1}^s \omega_{\pm, y}^{m_{l, 0}} \cdot f_{c|s}(x) \cdot \frac{d}{dx} (g_{k, m_l, a_l}(x))$$

we have that  $\frac{d^{k+1}}{dx^{k+1}} f_{c|s}(x) = \Sigma_1(x) + \Sigma_2(x)$ .

Next we analyze  $\Sigma_1(x)$  and  $\Sigma_2(x)$ . First, by eq. (A.2) we see that

$$\Sigma_1(x) = \sum_{l=1}^s \omega_{\pm,y}^{m_{l,0}} \cdot f_{c|s}(x) \cdot \xi_1(x) \cdot g_{k,m_l,a_l}(x) = \sum_{l=1}^s \omega_{\pm,y}^{m_{l,0}} \cdot f_{c|s}(x) \cdot \xi_1(x) \cdot \prod_{j=1}^k a_{l,j} \cdot \xi_j^{m_{l,j}}(x)$$

and if we denote  $\tilde{m}_{l,1} = m_{l,1} + 1$  and  $\forall_{2 \leq j \leq k} \tilde{m}_{l,j} = m_{l,j}$  we have

$$(A.6) \quad \Sigma_1(x) = \sum_{l=1}^s \omega_{\pm,y}^{m_{l,0}} \cdot f_{c|s}(x) \cdot \xi_1 \cdot \prod_{j=1}^k a_{l,j} \cdot \xi_j^{\tilde{m}_{l,j}}(x) = \sum_{l=1}^s \omega_{\pm,y}^{m_{l,0}} \cdot f_{c|s}(x) \cdot \xi_1(x) \cdot g_{k,\tilde{m}_l,a_l}(x).$$

Moving on to  $\Sigma_2(x)$ , we analyze  $\frac{d}{dx}(g_{k,m_l,a_l}(x))$  using General Leibniz rule [57]:

$$\begin{aligned} \frac{d}{dx}(g_{k,m_l,a_l}(x)) &= \frac{d}{dx} \left( \prod_{j=1}^k a_{l,j} \cdot \xi_j^{m_{l,j}}(x) \right) \\ &= \sum_{t_1+t_2+\dots+t_k=1} \left( \prod_{1 \leq s \leq k} \frac{d^{t_s}}{dx^{t_s}} (a_{l,s} \cdot \xi_s^{m_{l,s}}(x)) \right). \end{aligned}$$

Since  $t_s \in \mathbb{N}$  for each  $1 \leq s \leq k$  the equation  $t_1 + t_2 + \dots + t_k = 1$  implies that there are only  $k$  possibilities, where for some  $i = 1, \dots, k$  we have  $t_i = 1$  and for  $s \neq i$  we have  $t_s = 0$ . This leads to

$$\begin{aligned} &\sum_{t_1+t_2+\dots+t_k=1} \left( \prod_{1 \leq s \leq k} \frac{d^{t_s}}{dx^{t_s}} (a_{l,s} \cdot \xi_s^{m_{l,s}}(x)) \right) = \\ &m_{l,1} \cdot \xi_1^{-1}(x) \cdot \xi_2(x) \cdot \prod_{j=3}^k a_{l,j} \cdot \xi_j^{m_{l,j}}(x) + m_{l,2} \cdot \xi_2^{-1}(x) \cdot \xi_3(x) \cdot \prod_{\substack{j=1 \\ j \neq 2,3}}^k a_{l,j} \cdot \xi_j^{m_{l,j}}(x) + \dots \\ &+ m_{l,k-1} \cdot \xi_{k-1}^{-1}(x) \cdot \xi_k(x) \cdot \prod_{\substack{j=1 \\ j \neq k,k+1}}^k a_{l,j} \cdot \xi_j^{m_{l,j}}(x) + m_{l,k} \cdot \xi_k^{-1}(x) \cdot \xi_{k+1}(x) \cdot \prod_{j=1}^k a_{l,j} \cdot \xi_j^{m_{l,j}}(x). \end{aligned}$$

In each of the first  $k-1$  elements in the sum above the power of  $\xi_i$  is reduced by 1 and the power of  $\xi_{i+1}$  is increased by 1 where  $1 \leq i \leq k-1$  and in the last element the only power of  $\xi_k$  is reduced by 1 and there's a new element in the multiplication which is  $\xi_{k+1}$  with power and coefficient being both 1z, so to continue we will denote the powers of  $\xi_i$  in each element  $1 \leq j \leq k$  by  $\tilde{m}_{l_j,i}$ . We also have a similar change in the coefficients  $a_{l,j}$ , where each element  $j$  of the sum above is multiplied by the corresponding power of  $\xi_i$  so we denote the new coefficients by  $\tilde{a}_{l_j,i}$  and write:

$$\begin{aligned} &m_{l,1} \cdot \xi_1^{-1}(x) \cdot \xi_2(x) \cdot \prod_{j=3}^k a_{l,j} \cdot \xi_j^{m_{l,j}}(x) + m_{l,2} \cdot \xi_2^{-1}(x) \cdot \xi_3(x) \cdot \prod_{\substack{j=1 \\ j \neq 2,3}}^k a_{l,j} \cdot \xi_j^{m_{l,j}}(x) + \dots + \\ &m_{l,k-1} \cdot \xi_{k-1}^{-1}(x) \cdot \xi_k(x) \cdot \prod_{\substack{j=1 \\ j \neq k,k+1}}^k a_{l,j} \cdot \xi_j^{m_{l,j}}(x) + m_{l,k} \cdot \xi_k^{-1}(x) \cdot \xi_{k+1}(x) \cdot \prod_{j=1}^k a_{l,j} \cdot \xi_j^{m_{l,j}}(x) = \\ &\prod_{j=1}^k \tilde{a}_{l_1,j} \cdot \xi_j^{\tilde{m}_{l_1,j}}(x) + \prod_{j=1}^k \tilde{a}_{l_2,j} \cdot \xi_j^{\tilde{m}_{l_2,j}}(x) + \dots + \prod_{j=1}^k \tilde{a}_{l_{k-1},j} \cdot \xi_j^{\tilde{m}_{l_{k-1},j}}(x) + \prod_{j=1}^{k+1} \tilde{a}_{l_k,j} \cdot \xi_j^{\tilde{m}_{l_k,j}}(x) = \\ &g_{k,\tilde{a}_{l_1,j},\tilde{m}_{l_1,j}}(x) + g_{k,\tilde{a}_{l_2,j},\tilde{m}_{l_2,j}}(x) + \dots + g_{k,\tilde{a}_{l_{k-1},j},\tilde{m}_{l_{k-1},j}}(x) + g_{k+1,\tilde{a}_{l_k,j},\tilde{m}_{l_k,j}}(x) = \\ &\sum_{i=1}^{k-1} g_{k,\tilde{a}_{l_i,j},\tilde{m}_{l_i,j}}(x) + g_{k+1,\tilde{a}_{l_k,j},\tilde{m}_{l_k,j}}(x). \end{aligned}$$

Going back to eq. (A.5) we conclude that

$$\begin{aligned}
\Sigma_2(x) &= \sum_{l=1}^s \omega_{\pm,y}^{m_{l,0}} \cdot f_{c|s}(x) \cdot \left( \sum_{i=1}^{k-1} g_{k,\tilde{a}_{l_i,j},\tilde{m}_{l_i,j}}(x) + g_{k+1,\tilde{a}_{l_k,j},\tilde{m}_{l_k,j}}(x) \right) = \\
\text{(A.7)} \quad & \sum_{l=1}^s \omega_{\pm,y}^{m_{l,0}} \cdot f_{c|s}(x) \cdot \left( \sum_{i=1}^{k-1} g_{k,\tilde{a}_{l_i,j},\tilde{m}_{l_i,j}}(x) \right) + \sum_{l=1}^s \omega_{\pm,y}^{m_{l,0}} \cdot f_{c|s}(x) \cdot g_{k+1,\tilde{a}_{l_k,j},\tilde{m}_{l_k,j}}(x) = \\
& \sum_{l=1}^s \sum_{i=1}^{k-1} \omega_{\pm,y}^{m_{l,0}} \cdot f_{c|s}(x) \cdot g_{k,\tilde{a}_{l_i,j},\tilde{m}_{l_i,j}}(x) + \sum_{l=1}^s \omega_{\pm,y}^{m_{l,0}} \cdot f_{c|s}(x) \cdot g_{k+1,\tilde{a}_{l_k,j},\tilde{m}_{l_k,j}}(x)..
\end{aligned}$$

Finally, using eqs. (A.6) and (A.7) we have:

$$\begin{aligned}
\frac{d^{k+1}}{dx^{k+1}} f_{c|s}(x) &= \Sigma_1(x) + \Sigma_2(x) \\
&= \sum_{l=1}^s \omega_{\pm,y}^{m_{l,0}} \cdot f_{c|s}(x) \cdot \xi_1 \cdot g_{k,\tilde{m}_l,a_l}(x) \\
&\quad + \sum_{l=1}^s \sum_{i=1}^{k-1} \omega_{\pm,y}^{m_{l,0}} \cdot f_{c|s}(x) \cdot g_{k,\tilde{a}_{l_i,j},\tilde{m}_{l_i,j}}(x) \\
&\quad + \sum_{l=1}^s \omega_{\pm,y}^{m_{l,0}} \cdot f_{c|s}(x) \cdot g_{k+1,\tilde{a}_{l_k,j},\tilde{m}_{l_k,j}}(x).
\end{aligned}$$

So we can conclude that there exists  $s_1 \in \mathbb{N}$  s.t.:

$$\frac{d^{k+1}}{dx^{k+1}} f_{c|s}(x) = \sum_{l=1}^{s_1} \omega_{\pm,y}^{m_{l,0}} \cdot f_{c|s}(x) \cdot g_{k,m_l,a_l}(x).$$

Since

$$\begin{aligned}
\frac{d^{k+1}}{dx^{k+1}} f_{c|s}(x) &= \frac{d}{dx} \left( \frac{d^k}{dx^k} f_{c|s}(x) \right) = \frac{d}{dx} \left( \sum_{l=1}^s \omega_{\pm,y}^{m_{l,0}} \cdot f_{c|s}(x) \cdot g_{k,m_l,a_l}(x) \right) = \\
& \frac{d}{dx} \left( \omega_{\pm,y}^{m_{1,0}} \cdot f_{c|s}(x) \cdot g_{k,m_1,a_1}(x) + \sum_{l=2}^s \omega_{\pm,y}^{m_{l,0}} \cdot f_{c|s}(x) \cdot g_{k,m_l,a_l}(x) \right) = \\
& \frac{d}{dx} \left( \omega_{\pm,y}^{m_{1,0}} \cdot f_{c|s}(x) \cdot g_{k,m_1,a_1}(x) \right) + \frac{d}{dx} \left( \sum_{l=2}^s \omega_{\pm,y}^{m_{l,0}} \cdot f_{c|s}(x) \cdot g_{k,m_l,a_l}(x) \right) = \\
& \omega_{\pm,y}^{m_{1,0}} \cdot f_{c|s}(x)^{(1)} \cdot g_{k,m_1,a_1}(x) + \omega_{\pm,y}^{m_{1,0}} \cdot f_{c|s}(x) \cdot \frac{d}{dx} (g_{k,m_1,a_1}(x)) + \frac{d}{dx} \left( \sum_{l=2}^s \omega_{\pm,y}^{m_{l,0}} \cdot f_{c|s}(x) \cdot g_{k,m_l,a_l}(x) \right) = \\
& \omega_{\pm,y}^{m_{1,0}+1} \cdot a_{1,1} \cdot \xi_1^{m_{1,1}+1}(x) \cdot \prod_{j=2}^k a_{l,j} \cdot \xi_j^{m_{l,j}}(x) + \omega_{\pm,y}^{m_{1,0}} \cdot f_{c|s}(x) \cdot \frac{d}{dx} (g_{k,m_1,a_1}(x)) + \\
& \frac{d}{dx} \left( \sum_{l=2}^s \omega_{\pm,y}^{m_{l,0}} \cdot f_{c|s}(x) \cdot g_{k,m_l,a_l}(x) \right),
\end{aligned}$$

adding eq. (A.3) we have  $m_{1,0} + 1 = k + 1$  and  $m_{1,1} + 1 = k + 1$  which implies that the highest power of  $\omega_{\pm,y}$  in  $\frac{d^{k+1}}{dx^{k+1}} f_{c|s}(x)$  is  $k + 1$  and the highest power of  $\xi_i$  in  $\frac{d^{k+1}}{dx^{k+1}} f_{c|s}(x)$  is also  $k + 1$ .  $\square$

*Proof of Lemma 1.1.* First, since  $F_1, F_2, \xi$  are smooth in  $x$ , we can differentiate under the integral sign and conclude that  $\psi_{\omega_y}$  is smooth in  $x$ , with possibly a jump discontinuity at the endpoint  $x = -\pi$ . For an example of a situation with nonzero jump magnitudes, see Proposition 18 below.

Now let  $k \in \mathbb{N}$  s.t.  $1 \leq k \leq d$  and by using eq. (A.23) we get:

$$\text{(A.8)} \quad \frac{d^k}{dx^k} \psi_{\omega_y}(x) = \frac{1}{2\pi} \left( I_{\omega_y,k}(x) + \sum_{l=0}^{k-1} \frac{d^l}{dx^l} \left( e^{-\omega_y \xi(x)} \xi^{(1)}(x) A_{k-1-l}(x) \right) \right)$$

where (see eq. (A.20))

$$I_{\omega_y, k}(x) := \int_{-\pi}^{\xi(x)} e^{-\imath\omega_y y} \frac{\partial^{k+1}}{\partial x^{k+1}} F(x, y) dy + \int_{\xi(x)}^{\pi} e^{-\imath\omega_y y} \frac{\partial^{k+1}}{\partial x^{k+1}} F(x, y) dy$$

and since for each  $x \in \mathbb{T}$  we have  $F_x \in PC_{\mathbb{T}}^{(d+1, 1)}$  (recall definition 3), this implies that  $I_{\omega_y, k}$  is continuous in  $(-\pi, \pi)$  with one singularity at most at  $x = -\pi$ .

Now we proceed to analyze the sum on the right-hand side of eq. (A.8). First we begin by denoting  $\xi_i(x) := \frac{d^i}{dx^i} \xi(x)$ ,  $f_c(x) := \cos(\omega_y \xi(x))$ ,  $f_s(x) := \sin(\omega_y \xi(x))$ ,  $g_c(x) := f_c(x) \xi_1(x)$  and  $g_s(x) := f_s(x) \xi_1(x)$  so that we can claim that

$$\begin{aligned} \sum_{l=0}^{k-1} \frac{d^l}{dx^l} \left( e^{-\imath\omega_y \xi(x)} \xi_1(x) A_{k-1-l}(x) \right) &= \sum_{l=0}^{k-1} \frac{d^l}{dx^l} \left( (\cos(\omega_y \xi(x)) - \imath \sin(\omega_y \xi(x))) \xi_1(x) A_{k-1-l}(x) \right) \\ &= \sum_{l=0}^{k-1} \frac{d^l}{dx^l} \left( \cos(\omega_y \xi(x)) \xi_1(x) A_{k-1-l}(x) \right) \\ &\quad - \imath \sum_{l=0}^{k-1} \frac{d^l}{dx^l} \left( \sin(\omega_y \xi(x)) \xi_1(x) A_{k-1-l}(x) \right) \\ &= \sum_{l=0}^{k-1} \frac{d^l}{dx^l} \left( g_c(x) A_{k-1-l}(x) \right) - \imath \sum_{l=0}^{k-1} \frac{d^l}{dx^l} \left( g_s(x) A_{k-1-l}(x) \right). \end{aligned}$$

By eq. (A.1) we claim that

$$\begin{aligned} \sum_{l=0}^{k-1} \frac{d^l}{dx^l} \left( g_c(x) A_{k-1-l}(x) \right) - \imath \sum_{l=0}^{k-1} \frac{d^l}{dx^l} \left( g_s(x) A_{k-1-l}(x) \right) &= \\ \sum_{l=0}^{k-1} \sum_{i=0}^l a_i \cdot g_c^{(i)}(x) \cdot A_{k-1-l}^{(l-i)}(x) - \imath \sum_{l=0}^{k-1} \sum_{i=0}^l b_i \cdot g_s^{(i)}(x) \cdot A_{k-1-l}^{(l-i)}(x) &= \\ \sum_{l=0}^{k-1} \sum_{i=0}^l \left( a_i \cdot g_c^{(i)}(x) - i \cdot b_i \cdot g_s^{(i)}(x) \right) \cdot A_{k-1-l}^{(l-i)}(x) \end{aligned}$$

and furthermore by Lemma A.1

$$\begin{aligned} \sum_{l=0}^{k-1} \sum_{i=0}^l \left( a_i \cdot g_c(x)^{(i)} - i \cdot b_i \cdot g_s(x)^{(i)} \right) \cdot A_{k-1-l}^{(l-i)}(x) &= \\ \sum_{l=0}^{k-1} \sum_{i=0}^l \sum_{j=0}^i \left( a_i \cdot f_c^{(j)}(x) - i \cdot b_i \cdot f_s^{(j)}(x) \right) \cdot \xi_{i+1-j}(x) \cdot A_{k-1-l}^{(l-i)}(x). \end{aligned}$$

Adding Lemma A.2 we conclude that for  $k \in \mathbb{N}$  each of the following derivatives  $f_{c|s}^{(k)}(x)$  is a sum of the functions of the form

$$(A.9) \quad a_i \cdot \omega_{\pm, y}^i \cdot f_{c|s}(x) \cdot \xi_1^{m_1}(x) \cdots \xi_k^{m_n}(x)$$

where  $a_i \in \mathbb{R}$  and  $m_1, \dots, m_n \in \mathbb{N}$ . Then by combining all of our conclusions above, the following sum

$$(A.10) \quad \sum_{l=0}^{k-1} \frac{d^l}{dx^l} \left( e^{-\imath\omega_y \xi(x)} \xi^{(1)}(x) A_{k-1-l}(x) \right) =$$

$$(A.11) \quad \sum_{l=0}^{k-1} \sum_{i=0}^l \sum_{j=0}^i \left( a_i \cdot f_c^{(j)}(x) - i \cdot b_i \cdot f_s^{(j)}(x) \right) \cdot \xi_{i+1-j}(x) \cdot A_{k-1-l}^{(l-i)}(x)$$

is seen to be a sum of functions described in eq. (A.9) where the derivative order of  $\xi(x)$  is at most  $d_{\omega_y}$  which is continuous in  $(-\pi, \pi)$  with one singularity at most at  $x = -\pi$ . Then in turn, we have

$$\frac{d^k}{dx^k} \psi_{\omega_y}(x) = \frac{1}{2\pi} \left( I_{\omega_y, k}(x) + \sum_{l=0}^{k-1} \frac{d^l}{dx^l} \left( e^{-\imath\omega_y \xi(x)} \xi^{(1)}(x) A_{k-1-l}(x) \right) \right)$$



as a sum of continuous functions in  $(-\pi, \pi)$  with one singularity at most at  $x = -\pi$  for each  $1 \leq k \leq d$ .  $\square$

*Proof of Proposition 6.* The Bernoulli polynomials satisfy [64]:

$$\begin{cases} B_n(0) = B_n(1) & \text{if } n \neq 1 \\ B_n(0) = -B_n(1) = -\frac{1}{2} & \text{if } n = 1 \end{cases}, \quad \text{for } n \in \mathbb{N}$$

implying that:

$$(A.12) \quad \begin{cases} V_{\psi_{\omega_y}, l}(-\pi) - V_{\psi_{\omega_y}, l}(\pi) = 1 & \text{if } l = 0 \\ V_{\psi_{\omega_y}, l}(-\pi) - V_{\psi_{\omega_y}, l}(\pi) = 0 & \text{if } 1 \leq l \leq d_{\omega_y} \end{cases}.$$

Adding Lemma A.3 we have:

$$(A.13) \quad \frac{d^m}{dx^m} \phi_{\omega_y}(x)|_{x=-\pi} - \frac{d^m}{dx^m} \phi_{\omega_y}(x)|_{x=\pi} = \begin{cases} A_m^\psi(\omega_y) & \text{if } 0 \leq m \leq d_{\omega_y} \\ 0 & \text{if } m > d_{\omega_y} \end{cases}$$

so now we go back to  $c_k(\phi_{\omega_y}) := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} \phi_{\omega_y}(x) dx$  and use integration by parts with (A.13):

$$\begin{aligned} 2\pi c_k(\phi_{\omega_y}) &= \left[ \frac{e^{-ikx} \phi_{\omega_y}(x)}{-ik} \right]_{x=-\pi}^{x=\pi} + \frac{1}{ik} \int_{-\pi}^{\pi} e^{-ikx} \phi_{\omega_y}^{(1)}(x) dx \\ &= \frac{(-1)^k (\phi_{\omega_y}(-\pi) - \phi_{\omega_y}(\pi))}{ik} + \frac{1}{ik} \int_{-\pi}^{\pi} e^{-ikx} \phi_{\omega_y}^{(1)}(x) dx \\ &= \frac{(-1)^k A_0^\psi(\omega_y)}{ik} + \frac{1}{ik} \int_{-\pi}^{\pi} e^{-ikx} \phi_{\omega_y}^{(1)}(x) dx = \dots \\ &\dots = \sum_{l=0}^d \frac{(-1)^k A_l^\psi(\omega_y)}{(ik)^{l+1}}. \end{aligned}$$

As we defined  $c_0(\phi_{\omega_y}) \equiv 0$ , we conclude:

$$c_k(\phi_{\omega_y}) = \begin{cases} 0 & \text{if } k = 0 \\ \frac{(-1)^k}{2\pi} \sum_{l=0}^{d_{\omega_y}} \frac{A_l^\psi(\omega_y)}{(ik)^{l+1}} & \text{otherwise} \end{cases}. \quad \square$$

**Lemma A.3.** Let  $\omega_y \in \mathbb{Z}$  and let  $\phi_{\omega_y}(x)$  be as defined in eq. (2.3) a piecewise polynomial of degree  $d_{\omega_y}$ . Then:

$$(A.14) \quad \frac{d^m}{dx^m} \phi_{\omega_y}(x) = \begin{cases} -\frac{A_{m-1}^\psi(\omega_y)}{2\pi} + \sum_{l=m}^{d_{\omega_y}} A_l(\omega_y) V_{\psi, l-m}(x) & \text{if } 1 \leq m \leq d_{\omega_y} \\ -\frac{A_{d_{\omega_y}}^\psi(\omega_y)}{2\pi} & \text{if } m = d_{\omega_y} + 1 \\ 0 & \text{if } m > d_{\omega_y} + 1 \end{cases}$$

*Proof.* First we remark that for Bernoulli polynomials,  $B_n(x)$ , we have  $\frac{d}{dx} B_n(x) = n \cdot B_{n-1}(x)$  which in turn for  $m \leq n$  gives us:

$$\frac{d^m}{dx^m} B_{l+1}(x) = \prod_{k=0}^{m-1} (l - k + 1) \cdot B_{l-m}(x).$$

Therefore we have:

$$(A.15) \quad \frac{d^m}{dx^m} B_{l+1} \left( \frac{x+\pi}{2\pi} \right) = \begin{cases} \frac{\prod_{k=0}^{m-1} (l-k+1)}{(2\pi)^m} \cdot B_{l-m+1} \left( \frac{x+\pi}{2\pi} \right) & \text{if } m \leq l+1 \\ 0 & \text{if } m > l+1 \end{cases}$$

Now we have:

$$\begin{aligned}
\frac{d^m}{dx^m} V_{\psi,l}(x) &= \frac{d^m}{dx^m} \left( -\frac{(2\pi)^l}{(l+1)!} B_{l+1} \left( \frac{x+\pi}{2\pi} \right) \right) = -\frac{(2\pi)^l}{(l+1)!} \frac{d^m}{dx^m} \left( B_{l+1} \left( \frac{x+\pi}{2\pi} \right) \right) = \\
&= -\frac{(2\pi)^l}{(l+1)!} \begin{cases} \frac{\prod_{k=0}^{m-1} (l-k+1)}{(2\pi)^m} \cdot B_{l-m+1} \left( \frac{x+\pi}{2\pi} \right) & \text{if } m < l+1 \\ -\frac{1}{2\pi} & \text{if } m = l+1 \\ 0 & \text{if } m > l+1 \end{cases} \\
&= \begin{cases} \frac{(2\pi)^{l-m}}{(l-m+1)!} \cdot B_{l-m+1} \left( \frac{x+\pi}{2\pi} \right) & \text{if } m < l+1 \\ -\frac{1}{2\pi} & \text{if } m = l+1 \\ 0 & \text{if } m > l+1 \end{cases} = \begin{cases} V_{\psi,l-m}(x) & \text{if } m < l+1 \\ -\frac{1}{2\pi} & \text{if } m = l+1 \\ 0 & \text{if } m > l+1 \end{cases}
\end{aligned}$$

and if we denote  $k = l - m$  then

$$(A.16) \quad \frac{d^m}{dx^m} V_{\psi,l}(x) = \begin{cases} 0 & \text{if } k < -1 \\ -\frac{1}{2\pi} & \text{if } k = -1 \\ V_{\psi,k}(x) & \text{if } k > -1 \end{cases}.$$

Going back to  $\frac{d^m}{dx^m} \phi_{\omega_y}(x)$  using (A.16) and keeping in mind that  $m \in \mathbb{N}$  we get:

$$\begin{aligned}
\frac{d^m}{dx^m} \phi_{\omega_y}(x) &= \frac{d^m}{dx^m} \left( \sum_{l=0}^{d_{\omega_y}} A_l^\psi(\omega_y) V_{\psi,l}(x) \right) = \sum_{l=0}^{d_{\omega_y}} A_l^\psi(\omega_y) \frac{d^m}{dx^m} V_{\psi,l}(x) = \\
&= \sum_{l=0}^{d_{\omega_y}} A_l^\psi(\omega_y) \begin{cases} V_{\psi,l-m}(x) & \text{if } m < l+1 \\ -\frac{1}{2\pi} & \text{if } m = l+1 \\ 0 & \text{if } m > l+1 \end{cases} \\
&= \sum_{k=-m}^{d_{\omega_y}-m} A_{k+m}^\psi(\omega_y) \begin{cases} V_{\psi,k}(x) & \text{if } k > -1 \\ -\frac{1}{2\pi} & \text{if } k = -1 \\ 0 & \text{if } k < -1 \end{cases} \\
&= \sum_{k=-1}^{d_{\omega_y}-m} A_{k+m}^\psi(\omega_y) \begin{cases} V_{\psi,k}(x) & \text{if } k > -1 \\ -\frac{1}{2\pi} & \text{if } k = -1 \end{cases} \\
&= -\frac{A_{m-1}^\psi(\omega_y)(\omega_y)}{2\pi} + \sum_{k=0}^{d_{\omega_y}-m} A_{k+m}^\psi(\omega_y) V_{\psi,k}(x)
\end{aligned}$$

Switching back from  $k = l - m$  to  $l = m + k$  we conclude the proof:

$$\frac{d^m}{dx^m} \phi_{\omega_y}(x) = \begin{cases} -\frac{A_{m-1}^\psi(\omega_y)}{2\pi} + \sum_{l=m}^{d_{\omega_y}} A_l^\psi(\omega_y) V_{\psi,l-m}(x) & \text{if } 1 \leq m \leq d_{\omega_y} \\ -\frac{A_{d_{\omega_y}}^\psi(\omega_y)}{2\pi} & \text{if } m = d_{\omega_y} + 1 \\ 0 & \text{if } m > d_{\omega_y} + 1 \end{cases}. \quad \square$$

**A.2. A bound on the derivatives.** Here we prove a key technical estimate, used in the proof of Lemma 2.1.

**Proposition 15.** *Assume  $\omega_y \in \mathbb{Z}$  and let  $\gamma_{\omega_y}$  be as in eq. (2.2) so that  $\gamma_{\omega_y} \in C_{\mathbb{T}}^{d+1}$ ,  $d+1 \in \mathbb{N}$  and let  $k \leq d$ ,  $x \in \mathbb{T}$ . Then exists constants  $B_F$ ,  $A_x \geq 0$  (see definition 4) and  $A_{d,M_\xi}$  as defined in eq. (A.25) such that:*

$$\left| \frac{d^{k+1}}{dx^{k+1}} (\gamma_{\omega_y}) \right| \leq (2d+5) \left( B_F + A_{d,M_\xi} A_x |\omega_y|^d \right).$$

Before we prove Proposition 15 we develop some auxiliary statements.

**Lemma A.4.** *Let  $\xi(x)$  be as described in definitions 3 and 4 and let  $\omega_y \in \mathbb{Z}$ , and  $k \in \mathbb{N}$  such that  $k \leq d$ . Then there exists  $c_k \in \mathbb{N} \setminus \{0\}$  such that:*

$$(A.17) \quad \left| \frac{d^k}{dx^k} \left( e^{-i\omega_y \xi(x)} \right) \right| \leq c_k |\omega_y|^k M_\xi^{k^2}.$$

*Proof.* First we notice that

$$(A.18) \quad \begin{aligned} \left| \frac{d^k}{dx^k} \left( e^{-i\omega_y \xi(x)} \right) \right| &= \left| \frac{d^k}{dx^k} \left( \cos(\omega_y \xi(x)) - i \sin(\omega_y \xi(x)) \right) \right| \\ &\leq \left| \frac{d^k}{dx^k} \cos(\omega_y \xi(x)) \right| + \left| \frac{d^k}{dx^k} \sin(\omega_y \xi(x)) \right| \end{aligned}$$

and now we will look for a bound over  $|\cos(\omega_y \xi(x))|$  and  $|\sin(\omega_y \xi(x))|$ . Before we begin the proof let us note that since we are only interested in finding a bound over the two above functions, we will allow ourself to avoid finding the explicit formula for  $\frac{d^k}{dx^k} \cos(\omega_y \xi(x))$  and  $\frac{d^k}{dx^k} \sin(\omega_y \xi(x))$ , and because of the relation between  $\frac{d}{dx} \cos(x)$  and  $\frac{d}{dx} \sin(x)$  we will be using a more loose notations and assume that the bound over  $|\cos(\omega_y \xi(x))|$  will apply to  $\frac{d^k}{dx^k} \sin(\omega_y \xi(x))$  as well.

We begin borrowing specific notations from Lemma A.1:

- $\xi_i(x) := \frac{d^i}{dx^i} \xi(x)$
- $\omega_{\pm, y}^a := -\omega_y^a$  or  $\omega_y^a$ , where  $a \in \mathbb{N}$
- $f_{c|s}^{(k)}(x) := \frac{d^k}{dx^k} (\cos(\omega_y \xi(x)))$  or  $\frac{d^k}{dx^k} (\sin(\omega_y \xi(x)))$

Before adding another notations we observe the result of Lemma A.2 and conclude that for each derivative  $\frac{d^l}{dx^l} f_{c|s}(x)$  we will have a sum of equations such as  $\omega_{\pm, y}^i f_{c|s}(x) (\xi_1^{m_1}(x) \cdots \xi_k^{m_k}(x))$ , but if we are interested in finding a bound for  $\left| \frac{d^s}{dx^s} (e^{-i\omega_y \xi(x)}) \right|$  we will evaluate the absolute value of every equation of the type presented above. So together with the assumption that for every  $k \leq d_\xi$  we have a constant  $M_\xi$  such that  $|\xi(x)^k| \leq M_\xi$  and  $|f_{c|s}(x)| \leq 1$  we can claim that we can ignore the derivative order of  $\xi(x)$  and focus only on its power. Now we denote  $\xi_*(x) := \xi_k(x)$  for every  $0 \leq k \leq s$  and observe that

$$\begin{aligned} \left| \frac{d}{dx} (\xi_{(i)}(x))^a \right| &= \left| \frac{d}{dx} (\xi_*^a(x)) \right| \\ &= \left| a (\xi_{(i-1)}(x))^{a-1} \xi_{(1)}(x) \right| = \left| a \xi_*^{(a-1)}(x) \xi_*(x) \right| \\ &= a |\xi_*(x)|^a \leq a M_\xi^a \end{aligned}$$

Using Lemma A.2 we see that there exists  $s \in \mathbb{N}$  s.t.

$$\frac{d^k}{dx^k} f_{c|s}(x) = \sum_l^s \omega_{\pm, y}^{m_{l,0}} \cdot f_{c|s}(x) \cdot g_{k, m_l, a_l}(x).$$

Now we can proceed:

$$\begin{aligned} \left| \frac{d^k}{dx^k} f_{c|s}(x) \right| &= \left| \sum_{l=1}^s \omega_{\pm, y}^{m_{l,0}} \cdot f_{c|s}(x) \cdot g_{k, m_l, a_l}(x) \right| \\ &= \left| \sum_{l=1}^s \omega_{\pm, y}^{m_{l,0}} \cdot f_{c|s}(x) \cdot \left( \prod_{j=1}^k a_{l,j} \cdot \xi_j^{m_{l,j}}(x) \right) \right| \\ &\leq \sum_{l=1}^s |\omega_y|^{m_{l,0}} \cdot |f_{c|s}(x)| \cdot \left| \prod_{j=1}^k a_{l,j} \cdot \xi_j^{m_{l,j}}(x) \right| \\ &\stackrel{|f_{c|s}(x)| \leq 1}{\leq} \sum_{l=1}^s |\omega_y|^{m_{l,0}} \cdot \prod_{j=1}^k |a_{l,j}| \cdot |\xi_j(x)|^{m_{l,j}} \\ &\leq |\omega_y|^k \cdot \sum_{l=1}^s \prod_{j=1}^k |a_{l,j}| \cdot |\xi_*(x)|^{m_{l,j}} \\ &\leq |\omega_y|^k \cdot \sum_{l=1}^s \prod_{j=1}^k |a_{l,j}| \cdot |\xi_*(x)|^k \\ &\stackrel{|\xi_*(x)|^{m_{l,j}} \leq M_\xi^k}{\leq} |\omega_y|^k \cdot s \cdot M_\xi^{k^2} \cdot \sum_{l=1}^s \prod_{j=1}^k |a_{l,j}|. \end{aligned}$$

Denoting  $c_k := 2 \cdot s \cdot \sum_{l=1}^s \prod_{j=1}^k |a_{l,j}|$  together with eq. (A.18) implies

$$\left| \frac{d^k}{dx^k} \left( e^{-i\omega_y \xi(x)} \right) \right| \leq 2 \cdot \left| \frac{d^k}{dx^k} f_{c|s}(x) \right| \leq c_k \cdot |\omega_y|^k \cdot M_\xi^{k^2}$$

□

**Lemma A.5.** *Let  $x \in \mathbb{T}$  and  $s \in \mathbb{N}$ , take  $A_l(x)$  as defined in definition 4, then*

$$(A.19) \quad \left| \frac{d^s}{dx^s} \left( \xi^{(1)}(x) A_{s-l}(x) \right) \right| \leq 2^s M_\xi A_x.$$

*Proof.* Using General Leibniz rule [57] we have:

$$\begin{aligned} & \left| \frac{d^s}{dx^s} \left( \xi^{(1)}(x) A_{s-l}(x) \right) \right| = \left| \sum_{m=0}^s \binom{s}{m} \left( \xi^{(1)}(x) \right)^{(s-m)} \left( A_{s-l}(x) \right)^{(m)} \right| \\ & \leq \sum_{m=0}^s \binom{s}{m} \left| \left( \xi^{(1)}(x) \right)^{(s-m+1)} \right| \left| \left( A_{s-l}(x) \right)^{(m)} \right| \leq M_\xi A_x \sum_{m=0}^s \binom{s}{m} \leq 2^s M_\xi A_x. \end{aligned}$$

□

Now we denote

$$(A.20) \quad I_{\omega_y, k}(x) := \int_{-\pi}^{\xi(x)} e^{-i\omega_y y} \frac{\partial^{k+1}}{\partial x^{k+1}} F(x, y) dy + \int_{\xi(x)}^{\pi} e^{-i\omega_y y} \frac{\partial^{k+1}}{\partial x^{k+1}} F(x, y) dy$$

and formulate a useful result by applying Leibniz integral rule on  $I_{\omega_y, k}(x)$ :

$$(A.21) \quad \frac{d}{dx} \left( I_{\omega_y, k}(x) \right) = e^{-i\omega_y \xi(x)} \xi^{(1)}(x) A_k(x) + I_{\omega_y, k+1}(x).$$

**Proposition 16.** *Let  $x \in \mathbb{T}$ ,  $\omega_y \in \mathbb{Z}$  and let  $\psi_{\omega_y}$  as described in definition 5, then exists  $B_F \geq 0$  and  $A_{d, M_\xi}$  (as defined in eq. (A.25)) s.t. for every  $k = 0, 1, \dots, d$  we have*

$$(A.22) \quad \left| \frac{d^{k+1}}{dx^{k+1}} \psi_{\omega_y}(x) \right| \leq B_F + A_{d, M_\xi} \cdot A_x \cdot |\omega_y|^d$$

*Proof.*

$$\begin{aligned} 2\pi \frac{d^{k+1}}{dx^{k+1}} \psi_{\omega_y}(x) &= \frac{d^{k+1}}{dx^{k+1}} \left( I_{\omega_y, 0}(x) \right) = \frac{d^k}{dx^k} \left( e^{-i\omega_y \xi(x)} \xi^{(1)}(x) A_0(x) + I_{\omega_y, 1}(x) \right) \\ &= \frac{d^k}{dx^k} \left( I_{\omega_y, 1}(x) \right) + \frac{d^k}{dx^k} \left( e^{-i\omega_y \xi(x)} \xi^{(1)}(x) A_0(x) \right) \\ &= \frac{d^{k-1}}{dx^{k-1}} \left( e^{-i\omega_y \xi(x)} \xi^{(1)}(x) A_1(x) + I_{\omega_y, 2}(x) \right) + \frac{d^k}{dx^k} \left( e^{-i\omega_y \xi(x)} \xi^{(1)}(x) A_0(x) \right) \\ &= \frac{d^{k-1}}{dx^{k-1}} \left( I_{\omega_y, 2}(x) \right) + \sum_{l=k-1}^k \left( e^{-i\omega_y \xi(x)} \xi^{(1)}(x) A_{k-l}(x) \right) \\ &= \frac{d^{k-2}}{dx^{k-2}} \left( I_{\omega_y, 3}(x) \right) + \sum_{l=k-2}^k \left( e^{-i\omega_y \xi(x)} \xi^{(1)}(x) A_{k-l}(x) \right) \\ &= \frac{d^{k-3}}{dx^{k-3}} \left( I_{\omega_y, 4}(x) \right) + \sum_{l=k-3}^k \frac{d^l}{dx^l} \left( e^{-i\omega_y \xi(x)} \xi^{(1)}(x) A_{k-l}(x) \right) = \dots \\ &= \frac{d^{k-j}}{dx^{k-j}} \left( I_{\omega_y, j+1}(x) \right) + \sum_{l=k-j}^k \frac{d^l}{dx^l} \left( e^{-i\omega_y \xi(x)} \xi^{(1)}(x) A_{k-l}(x) \right) = \dots \\ &= I_{\omega_y, k+1}(x) + \sum_{l=0}^k \frac{d^l}{dx^l} \left( e^{-i\omega_y \xi(x)} \xi^{(1)}(x) A_{k-l}(x) \right). \end{aligned}$$

Therefore,

$$(A.23) \quad \frac{d^{k+1}}{dx^{k+1}} \psi_{\omega_y}(x) = \frac{1}{2\pi} \left( I_{\omega_y, k+1}(x) + \sum_{l=0}^k \frac{d^l}{dx^l} \left( e^{-i\omega_y \xi(x)} \xi^{(1)}(x) A_{k-l}(x) \right) \right).$$

Proceeding further, we have:

$$\begin{aligned}
\left| \frac{d^{k+1}}{dx^{k+1}} (\psi_{\omega_y}(x)) \right| &= \frac{1}{2\pi} \left| \left( I_{\omega_y, k+1}(x) + \sum_{l=0}^k \frac{d^l}{dx^l} \left( e^{-i\omega_y \xi(x)} \xi^{(1)}(x) A_{k-l}(x) \right) \right) \right| \\
&\leq \frac{1}{2\pi} \left( |I_{\omega_y, k+1}(x)| + \left| \sum_{l=0}^k \frac{d^l}{dx^l} \left( e^{-i\omega_y \xi(x)} \xi^{(1)}(x) A_{k-l}(x) \right) \right| \right) \\
&\leq \frac{1}{2\pi} \left( |I_{\omega_y, k+1}(x)| + \sum_{l=0}^k \left| \frac{d^l}{dx^l} \left( e^{-i\omega_y \xi(x)} \xi^{(1)}(x) A_{k-l}(x) \right) \right| \right).
\end{aligned}$$

By applying Leibniz General rule [57] on the sum above we further have

$$\left| \frac{d^{k+1}}{dx^{k+1}} (\psi_{\omega_y}(x)) \right| \leq \frac{1}{2\pi} |I_{\omega_y, k+1}(x)| + \frac{1}{2\pi} \sum_{l=0}^k \sum_{m=0}^l \binom{l}{m} \left| \frac{d^{l-m}}{dx^{l-m}} \left( e^{-i\omega_y \xi(x)} \right) \right| \left| \frac{d^m}{dx^m} \left( \xi^{(1)}(x) A_{k-l}(x) \right) \right|$$

and now using Lemmas A.4 and A.5 and denoting  $C_d := \max_{0 \leq k \leq d} \{ |c_k| \}$ , where  $c_k$  is defined in Lemma A.4 we get:

$$\begin{aligned}
\left| \frac{d^{k+1}}{dx^{k+1}} (\psi_{\omega_y}(x)) \right| &\leq \frac{1}{2\pi} |I_{\omega_y, k+1}(x)| + \frac{A_x M_\xi}{2\pi} \sum_{l=0}^k \sum_{m=0}^l \binom{l}{m} C_d 2^m |\omega_y|^{l-m} M_\xi^{(l-m)^2} \\
&\leq \frac{1}{2\pi} |I_{\omega_y, k+1}(x)| + \frac{A_x C_d M_\xi}{2\pi} \sum_{l=0}^k 2^l M_\xi^l \sum_{m=0}^l \binom{l}{m} (|\omega_y| M_\xi)^{l-m}
\end{aligned}$$

Using the known identities

$$\begin{aligned}
\sum_{k=0}^n \binom{n}{k} a^{n-k} &= (1+a)^n, \quad a \in \mathbb{R} \setminus \{0\} \\
\sum_{k=0}^n 2^k (1+a)^k &= \frac{2^{n+1} (1+a)^{n+1} - 1}{2a+1}
\end{aligned}$$

we proceed to obtain:

$$\begin{aligned}
\left| \frac{d^{k+1}}{dx^{k+1}} (\psi_{\omega_y}(x)) \right| &\leq \frac{1}{2\pi} |I_{\omega_y, k+1}(x)| + \frac{A_x C_d M_\xi}{2\pi} \sum_{l=0}^k (2M_\xi)^l (1 + |\omega_y| M_\xi)^l \\
&= \frac{1}{2\pi} |I_{\omega_y, k+1}(x)| + \frac{A_x C_d M_\xi}{2\pi} \frac{(2M_\xi)^{k+1} (1 + |\omega_y| M_\xi)^{k+1} - 1}{2|\omega_y| M_\xi + 1} \\
&\leq \frac{1}{2\pi} |I_{\omega_y, k+1}(x)| + \frac{A_x C_d M_\xi^{k+2}}{2\pi} \frac{2^{k+1} (1 + |\omega_y| M_\xi)^{k+1}}{1 + |\omega_y| M_\xi} \\
&= \frac{1}{2\pi} |I_{\omega_y, k+1}(x)| + \frac{A_x C_d M_\xi^{k+2}}{\pi} 2^k (1 + |\omega_y| M_\xi)^k \\
&\leq \frac{1}{2\pi} |I_{\omega_y, k+1}(x)| + \frac{A_x C_d M_\xi^{k+2}}{\pi} 2^k (2|\omega_y| M_\xi)^k \\
&= \frac{1}{2\pi} |I_{\omega_y, k+1}(x)| + \frac{A_x C_d}{\pi} 4^k M_\xi^{2k+2} |\omega_y|^k.
\end{aligned}$$

From definition 4 we have a constant  $B_F$  s.t.:

$$\text{(A.24)} \quad \left| \frac{\partial^{k+1}}{\partial x^{k+1}} (F(x, y)) \right| \leq B_F$$

which allows us to continue:

$$\begin{aligned}
|I_{\omega_y, k+1}| &\leq \int_{-\pi}^{\xi(x)} |e^{-iy\omega_y}| \left| \frac{\partial^{k+1}}{\partial x^{k+1}} F(x, y) \right| dy + \int_{\xi(x)}^{\pi} |e^{-iy\omega_y}| \left| \frac{\partial^{k+1}}{\partial x^{k+1}} F(x, y) \right| dy \\
&\leq \int_{-\pi}^{\xi(x)} \left| \frac{\partial^{k+1}}{\partial x^{k+1}} F(x, y) \right| dy + \int_{\xi(x)}^{\pi} \left| \frac{\partial^{k+1}}{\partial x^{k+1}} F(x, y) \right| dy \leq \int_{-\pi}^{\xi(x)} B_F dy + \int_{\xi(x)}^{\pi} B_F dy = 2\pi B_F.
\end{aligned}$$

Denoting

$$(A.25) \quad A_{d, M_\xi} := \frac{4^d C_d M_\xi^{2d+2}}{\pi}$$

we finally have

$$\begin{aligned} \left| \frac{d^{k+1}}{dx^{k+1}} (\psi_{\omega_y}(x)) \right| &\leq \frac{2\pi B_F}{2\pi} + \frac{A_x C_d}{\pi} 4^k M_\xi^{2k+2} |\omega_y|^k \\ &= B_F + \frac{A_x C_d}{\pi} 4^k M_\xi^{2k+2} |\omega_y|^k \\ &\stackrel{k \leq d}{\leq} B_F + \frac{A_x C_d}{\pi} 4^d M_\xi^{2d+2} |\omega_y|^d \\ &= B_F + A_{d, M_\xi} \cdot A_x \cdot |\omega_y|^d. \end{aligned} \quad \square$$

Before presenting Proposition 17 we need an additional technical result.

**Lemma A.6.** *Let  $\omega_y \in \mathbb{Z}$  and  $\psi_{\omega_y}$  as in definition 5. Then there exist constants  $B_F$  (as in definition 4) and  $A_{d, M_\xi}$  such that:*

$$(A.26) \quad \left| A_l^\psi(\omega_y) \right| \leq 2B_F + 2A_{d, M_\xi} A_x |\omega_y|^d.$$

*Proof.* From eq. (1.7) we get:

$$\left| A_l^\psi(\omega_y) \right| = \left| \psi_{\omega_y}^{(l)}(x) \Big|_{x=-\pi} - \psi_{\omega_y}^{(l)}(x) \Big|_{x=\pi} \right| \leq \left| \left( \psi_{\omega_y}^{(l)}(x) \Big|_{x=-\pi} \right) \right| + \left| \left( \psi_{\omega_y}^{(l)}(x) \Big|_{x=\pi} \right) \right|.$$

Using Proposition 16 and eq. (A.25) we have:

$$\begin{aligned} \left| A_l^\psi(\omega_y) \right| &\leq 2 \left( B_F + \frac{A_x C_d}{\pi} 4^d M_\xi^{2d+2} |\omega_y|^d \right) \\ &= 2 \left( B_F + A_{d, M_\xi} A_x |\omega_y|^d \right), \end{aligned}$$

completing the proof. □

**Proposition 17.** *Let  $\omega \in \mathbb{Z}$  and  $\phi_{\omega_y}$  as defined in eq. (2.3) and  $k \in \mathbb{N}$ , then*

$$(A.27) \quad \left| \frac{d^k}{dx^k} \phi_{\omega_y}(x) \right| \leq 2(d+2) \left( B_F + A_{d, M_\xi} A_x |\omega_y|^d \right),$$

where  $A_{d, M_\xi}$  is defined in eq. (A.25)

*Proof.*

$$\begin{aligned} \frac{d}{dx} \phi_{\omega_y}(x) &= \frac{d}{dx} \left( \sum_{l=0}^d A_l^\psi(\omega_y) V_{\psi, l}(x) \right) = \sum_{l=0}^d \frac{d}{dx} \left( A_l^\psi(\omega_y) V_{\psi, l}(x) \right) \\ &= \sum_{l=0}^d A_l^\psi(\omega_y) \frac{d}{dx} (V_{\psi, l}(x)) \\ &= \sum_{l=0}^d A_l^\psi(\omega_y) \frac{d}{dx} \left( -\frac{(2\pi)^l}{(l+1)!} B_{l+1} \left( \frac{x+\pi}{2\pi} \right) \right) \\ &= \sum_{l=0}^d A_l^\psi(\omega_y) \left( -\frac{(2\pi)^{l-1}}{(l)!} B_l \left( \frac{x+\pi}{2\pi} \right) \right) \\ &= \sum_{l=0}^d A_l^\psi(\omega_y) V_{\psi, l-1}(x) = -\frac{A_0^\psi(\omega_y)}{2\pi} + \sum_{l=1}^d A_l^\psi(\omega_y) V_{\psi, l-1}(x) \\ &= -\frac{A_0^\psi(\omega_y)}{2\pi} + \sum_{l=0}^{d-1} A_{l+1}^\psi(\omega_y) V_{\psi, l}(x). \end{aligned}$$

Computing the second derivative we have:

$$\begin{aligned}
\frac{d^2}{dx^2} \phi_{\omega_y}(x) &= \frac{d}{dx} \left( -\frac{A_0^\psi(\omega_y)}{2\pi} + \sum_{l=0}^{d-1} A_{l+1}^\psi(\omega_y) V_{\psi,l}(x) \right) \\
&= -\frac{A_1^\psi(\omega_y)}{2\pi} + \sum_{l=1}^{d-1} A_{l+1}^\psi(\omega_y) V_{\psi,l-1}(x) \\
&= -\frac{A_1^\psi(\omega_y)}{2\pi} + \sum_{l=0}^{d-2} A_{l+2}^\psi(\omega_y) V_{\psi,l}(x).
\end{aligned}$$

By induction we can show that:

$$\text{(A.28)} \quad \frac{d^k}{dx^k} \phi_{\omega_y}(x) = \begin{cases} -\frac{A_{k-1}^\psi(\omega_y)}{2\pi} + \sum_{l=0}^{d-k} A_{l+k}^\psi(\omega_y) V_{\psi,l}(x), & \text{if } k \leq d \\ -\frac{A_d^\psi(\omega_y)}{2\pi}, & \text{if } k = d+1 \\ 0, & \text{otherwise} \end{cases}$$

With the above we now can estimate

$$\begin{aligned}
\left| \frac{d^k}{dx^k} \phi_{\omega_y}(x) \right| &\leq \left| -\frac{A_{k-1}^\psi(\omega_y)}{2\pi} + \sum_{l=0}^{d-k} A_{l+k}^\psi(\omega_y) V_{\psi,l}(x) \right| \\
&\leq \frac{|A_{k-1}^\psi(\omega_y)|}{2\pi} + \sum_{l=0}^{d-k} |A_{l+k}^\psi(\omega_y)| |V_{\psi,l}(x)| \\
&\leq \frac{|A_{k-1}^\psi(\omega_y)|}{2\pi} + \sum_{l=0}^{d-k} |A_{l+k}^\psi(\omega_y)| \\
&\leq |A_{k-1}^\psi(\omega_y)| + \sum_{l=0}^{d-k} |A_{l+k}^\psi(\omega_y)| = \sum_{l=k-1}^d |A_l^\psi(\omega_y)|.
\end{aligned}$$

Using Lemma A.6 we conclude the proof:

$$\left| \frac{d^k}{dx^k} \phi_{\omega_y}(x) \right| \leq 2(d+2) \left( B_F + A_{d,M_\xi} A_x |\omega_y|^d \right). \quad \square$$

*Proof of Proposition 15.* From eq. (2.2) we have:

$$\frac{d^{k+1}}{dx^{k+1}} (\gamma_{\omega_y}) = \frac{d^{k+1}}{dx^{k+1}} (\psi_{\omega_y}) - \frac{d^{k+1}}{dx^{k+1}} (\phi_{\omega_y})$$

From Propositions 16 and 17 we have:

$$\begin{aligned}
\left| \frac{d^{k+1}}{dx^{k+1}} (\gamma_{\omega_y}) \right| &\leq \left| \frac{d^{k+1}}{dx^{k+1}} (\psi_{\omega_y}) \right| + \left| \frac{d^{k+1}}{dx^{k+1}} (\phi_{\omega_y}) \right| \leq \\
&\leq B_F + A_{d,M_\xi} A_x |\omega_y|^d + 2(d+2) \left( B_F + A_{d,M_\xi} A_x |\omega_y|^d \right) \cdot \\
&\leq (2d+5) \left( B_F + A_{d,M_\xi} A_x |\omega_y|^d \right)
\end{aligned}$$

Therefore,

$$\text{(A.29)} \quad \left| \frac{d^{k+1}}{dx^{k+1}} (\gamma_{\omega_y}) \right| \leq (2d+5) \left( B_F + A_{d,M_\xi} A_x |\omega_y|^d \right),$$

completing the proof of Proposition 15. □

**A.3. Condition for discontinuous  $\psi_{\omega_y}$ .** Now we demonstrate that  $\psi_{\omega_y}$  may generically have a discontinuity at the domain boundary ( $x = \pm\pi$ ). We focus on a simple example where the discontinuity curve is  $\xi(x) = x$ . Substitution into eq. (A.23) entails:

$$2\pi \frac{d^{k+1}}{dx^{k+1}} \psi_{\omega_y}(x) = \sum_{l=0}^k \frac{d^l}{dx^l} \left( e^{-i\omega_y x} A_{k-l}(x) \right) + \int_{-\pi}^x e^{-i\omega_y y} \frac{\partial^{k+1}}{\partial x^{k+1}} F(x, y) dy + \int_x^{\pi} e^{-i\omega_y y} \frac{\partial^{k+1}}{\partial x^{k+1}} F(x, y) dy.$$

Using Leibniz general rule [57] and  $e^{-i\omega_y \pi} = e^{i\omega_y \pi} = (-1)^{\omega_y}$ ,  $\omega_y \in \mathbb{Z}$  we have:

$$2\pi \frac{d^{k+1}}{dx^{k+1}} \psi_{\omega_y}(x) = (-1)^{\omega_y} \sum_{l=0}^k \sum_{m=0}^l \binom{l}{m} (-i\omega_y)^{l-m} A_{k-l}^{(m)}(x) + \int_{-\pi}^x e^{-i\omega_y y} \frac{\partial^{k+1}}{\partial x^{k+1}} F(x, y) dy + \int_x^{\pi} e^{-i\omega_y y} \frac{\partial^{k+1}}{\partial x^{k+1}} F(x, y) dy.$$

Using eq. (1.7) we have:

$$\begin{aligned} 2\pi A_{k+1}^{\psi}(\omega_y) &= (-1)^{\omega_y} \sum_{l=0}^k \sum_{m=0}^l \binom{l}{m} (-i\omega_y)^{l-m} A_{k-l}^{(m)}(-\pi) \\ &\quad + \int_{-\pi}^{-\pi} e^{-i\omega_y y} \frac{\partial^{k+1}}{\partial x^{k+1}} F(x, y) \Big|_{x=-\pi} dy \\ &\quad - (-1)^{\omega_y} \sum_{l=0}^k \sum_{m=0}^l \binom{l}{m} (-i\omega_y)^{l-m} A_{k-l}^{(m)}(\pi) \\ &\quad - \int_{-\pi}^{\pi} e^{-i\omega_y y} \frac{\partial^{k+1}}{\partial x^{k+1}} F(x, y) \Big|_{x=\pi} dy \\ &= (-1)^{\omega_y} \sum_{l=0}^k \sum_{m=0}^l \binom{l}{m} (-i\omega_y)^{l-m} \left( A_l^{(m)}(-\pi) - A_l^{(m)}(\pi) \right) + \\ &\quad \int_{-\pi}^{\pi} e^{-i\omega_y y} \left( \frac{\partial^{k+1}}{\partial x^{k+1}} F(x, y) \Big|_{x=-\pi} - \frac{\partial^{k+1}}{\partial x^{k+1}} F(x, y) \Big|_{x=\pi} \right) dy. \end{aligned}$$

Denoting:

$$\frac{\partial^{k+1}}{\partial x^{k+1}} F(x, y) := F_x^{(k+1)}(x, y)$$

we get:

$$2\pi A_{k+1}^{\psi}(\omega_y) = (-1)^{\omega_y} \sum_{l=0}^k \sum_{m=0}^l \binom{l}{m} (-i\omega_y)^{l-m} (A_l(-\pi) - A_l(\pi)) + \int_{-\pi}^{\pi} e^{-i\omega_y y} \left( F_x^{(k+1)}(-\pi, y) - F_x^{(k+1)}(\pi, y) \right) dy.$$

Now if  $\psi_{\omega_y}$  has  $d_{\omega_y}$  derivatives then in order to have  $A_{k+1}^{\psi}(\omega_y) = 0$  for each  $0 \leq k \leq d_{\omega_y} - 1$  we must have:

$$\begin{aligned} &(-1)^{\omega_y} \sum_{l=0}^k \sum_{m=0}^l \binom{l}{m} (-i\omega_y)^{l-m} \left( A_l^{(m)}(-\pi) - A_l^{(m)}(\pi) \right) + \\ &\quad \int_{-\pi}^{\pi} e^{-i\omega_y y} \left( F_x^{(k+1)}(-\pi, y) - F_x^{(k+1)}(\pi, y) \right) dy = 0. \end{aligned}$$

If we assume that all derivatives  $F_x^l$ ,  $l = 0, \dots, l$  are periodic (in  $x$ ) for all  $y$ , then we have:

$$\int_{-\pi}^{\pi} e^{-i\omega_y y} \left( F_x^{(k+1)}(-\pi, y) - F_x^{(k+1)}(\pi, y) \right) dy = 0$$



which leaves us with:

$$A_k^\psi(\omega_y) = 0 \Leftrightarrow \sum_{l=0}^k \sum_{m=0}^l \binom{l}{m} (-i\omega_y)^{l-m} \left( A_l^{(m)}(-\pi) - A_l^{(m)}(\pi) \right) = 0.$$

Denoting:

$$\Delta_l^{(m)} := A_l^{(m)}(-\pi) - A_l^{(m)}(\pi)$$

we claim:

$$A_{k+1}^\psi(\omega_y) = 0 \Leftrightarrow \sum_{l=0}^k \sum_{m=0}^l \binom{l}{m} \Delta_l^{(m)} (-i\omega_y)^{l-m} = 0.$$

Also notice that:

$$\begin{aligned} A_1^\psi(\omega_y) &= \Delta_0^{(0)} \\ A_2^\psi(\omega_y) &= \Delta_0^{(0)} + \Delta_1^{(0)}(-i\omega_y) + \Delta_1^{(1)} = A_1^\psi(\omega_y) + \Delta_1^{(0)}(-i\omega_y) + \Delta_1^{(1)} \\ A_3^\psi(\omega_y) &= \Delta_0^{(0)} + \Delta_1^{(0)}(-i\omega_y) + \Delta_1^{(1)} + \Delta_2^{(0)}(-i\omega_y)^2 + \Delta_2^{(1)}(-i\omega_y) + \Delta_2^{(2)} \\ &= A_2^\psi(\omega_y) + \Delta_2^{(0)}(-i\omega_y)^2 + \Delta_2^{(1)}(-i\omega_y) + \Delta_2^{(2)} \\ &\vdots \\ A_{k+1}^\psi(\omega_y) &= A_k^\psi(\omega_y) + \sum_{m=0}^k \Delta_k^{(m)} \binom{k}{m} (-i\omega_y)^{k-m} \end{aligned}$$

So in order for  $A_k^\psi(\omega_y) = 0$  to take place we would have to demand that  $A_k^\psi(\omega_y) = -\sum_{m=0}^k \Delta_k^{(m)} \binom{k}{m} (-i\omega_y)^{k-m} = \sum_{m=0}^k \Delta_k^{(m)} (-1)^{k-m+1} \binom{k}{m} (-i\omega_y)^{k-m}$  and by denoting:

$$\begin{aligned} \vec{\Delta}_k &:= \left( \Delta_k^{(0)}, \Delta_k^{(1)}, \dots, \Delta_k^{(k-1)}, \Delta_k^{(k)} \right) \\ \vec{\Omega}_k &:= \left( (-1)^{k+1} \binom{k}{0} (i\omega_y)^k, (-1)^k \binom{k}{1} (i\omega_y)^{k-1}, \dots, (-1)^2 \binom{k}{k-1} (i\omega_y)^k, -1 \right) \end{aligned}$$

we finally write a necessary and sufficient condition for the absence of the boundary discontinuity of  $\psi_{\omega_y}$ :

$$A_{k+1}^\psi(\omega_y) = 0 \iff A_k^\psi(\omega_y) = \vec{\Delta}_k \cdot \vec{\Omega}_k.$$

Suppose  $A_0^\psi(\omega_y) = \dots = A_k^\psi(\omega_y) = 0$  then for  $A_{k+1}^\psi(\omega_y) = 0$  we need  $\vec{\Delta}_k \perp \vec{\Omega}_k$ . In more detail  $A_0^\psi(\omega_y) = 0$  implies  $A_1^\psi(\omega_y) = 0 \iff \vec{\Delta}_0 \perp \vec{\Omega}_0$  and then  $A_2^\psi(\omega_y) = 0 \iff \vec{\Delta}_1 \perp \vec{\Omega}_1$  and so on.

The above is summarized in the following proposition:

**Proposition 18.** *Let  $F(x, y)$  as in definition 3 and  $\psi_{\omega_y}$  as in eq. (1.6) and Lemma 1.1. Assume that  $\frac{d^k}{dx^k} F(x, y) \Big|_{x=x_0+2\pi n} = \frac{d^k}{dx^k} F(x, y) \Big|_{x=x_0}$ ,  $n \in \mathbb{Z}$ , and  $A_0(\omega_y) = 0$ . Then:*

$$A_k^\psi(\omega_y) = 0 \iff \vec{\Delta}_{k-1} \perp \vec{\Omega}_{k-1}, \quad k = 1, \dots, d_{\omega_y}.$$

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