Soft extrapolation of bandlimited functions

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***Abstract*—Soft extrapolation refers to the problem of recovering a function from its samples multiplied by a fast-decaying window – in this note a narrow Gaussian. The question is akin to deconvolution, but leverages smoothness of the function in order to achieve stable recovery over an interval potentially larger than the essential support of the window. In case the function is bandlimited, we provide an error bound for extrapolation by a least-squares polynomial fit of a well-chosen degree: it is (morally) proportional to a fractional power of the perturbation level, which goes from 1 near the available samples, to 0 when the extrapolation distance reaches the characteristic smoothness length scale of the function. This bound is minimax in the sense that no algorithm can yield a meaningfully lower error over the same smoothness class. The result in this note can be put in the context of blind superresolution, where it corresponds to the limit of a single spike corrupted by a compactly-supported blur.**

I. Introduction

The problem of (computational) super-resolution consists in recovering fine features of a signal from limited knowledge of its spectrum, when the limit is usually dictated by the measurement device (such as the diffraction limit in optics), and there exists some a-priori information regarding the signal [1]. Much recent activity (e.g. [2]–[5] and references therein) has focused on the case where this a priori information is sparsity in time/space, employing methods such as line spectral estimation or exponential fitting [6], or ℓ_1 minimization. Accurate recovery of the signal of time/space is equivalent to extrapolation in the Fourier domain, outside the band where it is observed.

The structural signal assumption in this paper is not sparsity in time/space, but narrow compact support. In the Fourier domain, this is equivalent to a strong smoothness condition: the Fourier transform should be bandlimited. Instead of considering that this Fourier transform is known in an interval – the hard case – we instead assume that it is only known via its (noisy, sampled) multiplication by a Gaussian window – the soft case. The "soft extrapolation" question is then to recover the original bandlimited function in a stable fashion, over a larger interval than suggested by the Gaussian window. For this question to have a favorable answer, the characteristic length scale of the Fourier transform (inverse of the support length in time/space) should be much larger than the width of the Gaussian.

Bandlimited signal extrapolation has been considered in numerous articles, e.g. the well-known Papoulis-Gerschberg iterative algorithm [7], [8] and its various generalizations [9], PSWFs [10], [11], sinc interpolation [12], and hybrid methods [11]; however we are not aware of (practical or theoretical) results involving "soft" windows. This note is a continuation of the line of work started in [13], where the authors considered extrapolation of analytic functions from their observation in a finite interval.

Assuming that the noise level is known, and that sufficiently many samples are available in the tails of the Gaussian, we show that a weighted polynomial least squares fit with a judiciously chosen truncation order provides accurate reconstruction even outside the essential support of the Gaussian window. The least-squares weight is taken to be the Gaussian window itself.

II. "Soft extrapolation"

A. Problem setup

Definition 1. Let S_σ denote the class of bandlimited functions with bandlimit $\sigma > 0$:

$$g(\omega) = \int_{-\sigma}^{\sigma} f(t)\, e^{\imath\omega t} dt. \tag{1}$$

Then by the Paley Wiener theorem, $g(\omega)$ can be extended to an entire function of exponential type σ, satisfying

$$|g(z)| \sim e^{\sigma|z|}, \quad |z| \to \infty.$$

Suppose that we want to recover $g \in S_\sigma$ as above from the knowledge of its corrupted and noisy samples on an equispaced grid

$$\widetilde{h}(\omega_k) = g(\omega_k)\, W(\omega_k) + e(\omega_k),$$
$$\omega_k = k\Delta\omega,\ k = 0, \pm 1, \pm 2, \ldots, \pm N,$$

where $|e(\omega_k)| < \epsilon$ is the measurement noise and

$$W(\omega) := e^{-\frac{\omega^2}{2}} \tag{2}$$

is the "soft" window (corresponding to a low-pass Gaussian filter in the time domain). Using the a priori knowledge of σ, ϵ, we wish to choose the parameters $N, \Delta\omega$ and construct a function $\widetilde{g}(\omega)$ such that the error

$$B_{\epsilon,\sigma}^{\tilde{g}}(\omega) := |g(\omega) - \widetilde{g}(\omega)|$$

is smallest possible, at least asymptotically for small ϵ.

Remark 1. The variance of the Gaussian window is chosen to be unity without loss of generality. Indeed, if $W_\tau(\omega) = \exp\left(-\frac{\omega^2}{2\tau^2}\right)$ for some $\tau > 0$ and $g \in S_\sigma$, then the rescaling

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$w' = \omega\tau$ brings us back to the original problem for $g_1(\omega) := g(\omega\tau) \in S_{\sigma\tau}$. The extrapolation question is only interesting when σ is small when compared to the width of the Gaussian.

Remark 2. Without loss of generality we assume $g(\omega)$ to be real-valued on the real axis. All the results are immediately extended to the general case by considering (extrapolating) the real and the imaginary parts separately.

B. Elementary bound

A naive approach would be to approximate $g(\omega)$ directly from $\widetilde{h}(\omega)\frac{1}{W(\omega)}$. Indeed, if $\epsilon = 0$ this provides an essentially optimal method, since for any target point ω_0 and target accuracy δ we can sample in $[-c\omega_0, c\omega_0]$ for large enough c and approximate the underlying function (by any method for analytic functions, such as [13]) to accuracy $\delta W(\omega_0)$ uniformly in $[-\omega_0, \omega_0]$, giving final error $|\widetilde{g}(\omega_0) - g(\omega_0)| \sim \delta$.

If $\epsilon > 0$, however, the noise gets multiplied by $W(\omega)$ as well, resulting in the error at best

$$B^{naive}_{\epsilon,\sigma}(\omega) \sim \frac{1}{W(\omega)}\epsilon,$$

which of course can be prohibitively large even for moderate values of ω. The question becomes: *can one obtain an essentially better result by choosing the sampling interval and the approximation method judiciously?*

III. Main result

Let $\mathcal{P}_M$ denote the space of algebraic polynomials of degree at most M. Given (σ, ϵ), our method is to construct a polynomial extrapolant $\tilde{g} = p_M \in \mathcal{P}_M$ to $g(\omega)$ by solving a weighted least squares problem using the noisy data $\tilde{h}(\underline{\omega})$ on the appropriately chosen sampling grid $\underline{\omega} = (-N\Delta\omega, \dots, N\Delta\omega)$, where the parameters $E = E(\sigma, \epsilon) := (M, N, \Omega := N\Delta\omega)$ are to be determined. The least squares problem reads

$$p_M = \arg\min_{p \in \mathcal{P}_M} \left\| \tilde{h}(\underline{\omega}) - p(\underline{\omega}) \right\|_W^2, \qquad (3)$$

where the weighted norm is defined by

$$\|\underline{x}\|_W^2 := \underline{x}^T \left[diag\left\{W(\omega_k)\right\}\right] \underline{x}.$$

The main result of this work is that the following choice $E_* = (M_*, N_*, \Omega_*)$ provides an optimal extrapolation for S_σ:

$$\begin{aligned} M_* &:= \left\lceil \frac{1}{q} \log \frac{1}{\epsilon} \right\rceil, \\ N_* &:= 2M_*, \\ \Omega_* &:= \sqrt{2M_* + 1}, \end{aligned}$$

where $q = q(\sigma, \epsilon)$ is a "smoothness exponent", defined in (12) below, which admits asymptotically

$$q(\sigma, \epsilon) \approx \begin{cases} \frac{1}{2} \log\log \frac{1}{\epsilon} & \epsilon \ll \exp\left(-\frac{1}{\sigma^2}\right) \ll 1 \\ \log \frac{1}{\sigma} & \epsilon > \exp\left(-\frac{1}{\sigma^2}\right). \end{cases} \qquad (4)$$

Our main result bounds the error $B^{\tilde{g}}_{\sigma,\epsilon}(\omega)$, and shows that it is asymptotically minimax. The outline of the proof is presented in Section V.

Theorem 1. *Let $\rho = \rho_*(M_*, N_*, \Omega_*)$ be as above, and let $\widetilde{g}(\omega) = p_{M^*}(\omega)$ be as defined by* (3)*.*

1) *The error $B^{\tilde{g}}_{\epsilon,\sigma}(\omega)$ satisfies, up to polynomial factors in N_*, M_*, the bound $B^{\tilde{g}}_{\epsilon,\sigma}(\omega) \lessapprox B^{opt}_{\sigma,\epsilon}(\omega)$ where*

$$B^{opt}_{\sigma,\epsilon}(\omega) := \begin{cases} \epsilon \exp\left(\frac{\omega^2}{2}\right) \leqslant \epsilon^{1-\frac{1}{q}} & |\omega| \leqslant \Omega_*, \\ \exp\left(\frac{\beta\omega^2}{2}\right) \epsilon^{1-\frac{1}{2q}\log\left(\frac{2}{\beta}-1\right)} & \Omega_* < |\omega| < \frac{M_*}{\sigma}, \\ \exp(\sigma|\omega|) & |\omega| \geqslant \frac{M_*}{\sigma}, \end{cases}$$

where $\beta = 1 - \sqrt{1 - \frac{2M_*+1}{\omega^2}}$.

2) *Furthermore, for every pair σ, ϵ there exists a function $g_{\sigma,\epsilon}(\omega) \in S_\sigma$ which has the same asymptotic growth as $B^{opt}_{\sigma,\epsilon}(\omega)$ above. In particular, $|g_{\sigma,\epsilon}(\omega) W(\omega)| < \epsilon$ for $|\omega| < \Omega_*$, and therefore the constant zero function is a valid extrapolant to $g_{\sigma,\epsilon}(\omega)$ at noise level ϵ using samples in $[-\Omega_*, \Omega_*]$.*

To ease digestion and understanding of the above bounds, let us consider the regime where $\epsilon \ll \exp\left(-\frac{1}{\sigma^2}\right)$. It can be shown that there are three distinct regions in ω with respect to the decay of the error as $\epsilon \to 0$:

1) the "approximation", or "sampling" region $|\omega| \leqslant \Omega_* \approx \sqrt{\log \frac{1}{\epsilon}}$, where by design the error $B \lessapprox \frac{1}{W(\omega)}\epsilon$ decays linearly in ϵ ;
2) the "extrapolation" region $\sqrt{\log \frac{1}{\epsilon}} \approx \Omega_* < |\omega| < \frac{1}{\sigma}M_* \approx \log \frac{1}{\epsilon}$, where
$$B(\omega) \lessapprox \epsilon^{2 - \frac{\log \omega^2}{\log\log \frac{1}{\epsilon}} + o(1)},$$
so that the exponent $\rho(\omega) := 2 - \frac{\log \omega^2}{\log\log \frac{1}{\epsilon}}$ changes from $2 - 1 = 1$ at the left boundary to $2 - 2 = 0$ at the right boundary (see Fig.1);
3) the "forbidden" region $|\omega| \geqslant \frac{1}{\sigma}M_* \approx \log \frac{1}{\epsilon}$ where $B \lessapprox \exp(\sigma|\omega|)$ and therefore essentially no information can be extracted (the constant zero extrapolant attains best possible asymptotic accuracy).

IV. A numerical demonstration

We demonstrate the method and the bounds from Section III on the example of extrapolating the function

$$g(\omega) = 5 + \cosh(\sigma\omega - 2) + \sinh(\sigma\omega). \qquad (5)$$

1) In Fig.2 we show the reconstruction and the corresponding errors + bounds for fixed σ, ϵ. As can be seen in Fig.2a, the derived bounds are reasonably accurate. In Fig.2b it is clearly seen that the algorithm chooses a reasonable value for M_*, avoiding the extreme noise outside the essential support of the window.
2) In Fig.3 the reconstruction was performed with fixed ω and σ, varying ϵ. It can be seen that the dependence of the error on ϵ is accurately determined. The threshold values of ϵ for which one moves from interpolation to extrapolation region ($\epsilon_{1\to2}$) and from extrapolation to forbidden region ($\epsilon_{2\to3}$) can be approximately determined as the solutions of the equations $\epsilon = \exp\left(-q\omega^2/2\right)$ and $\epsilon = \exp(-q\sigma\omega)$, respectively.

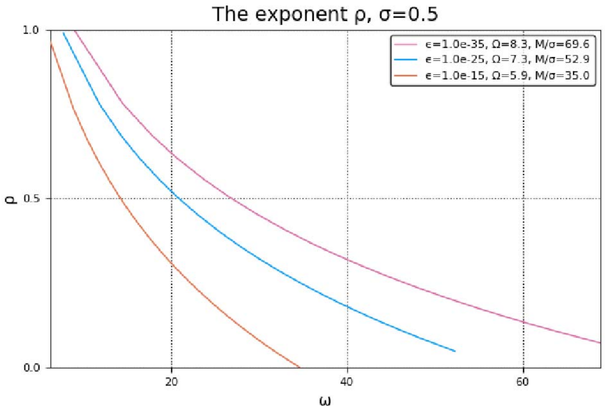


Fig. 1: An example of the function $\rho(\omega) = 2 - \frac{\log \omega^2}{\log \log \frac{1}{\epsilon}}$ for $\sigma = 0.5$ and $\epsilon = 10^{-35}, 10^{-25}, 10^{-15}$. As can be seen, at the left boundary ($\omega_{1\to 2} = \Omega_*$) the exponent is $\rho_{1\to 2} = 2-1 = 1$, while at the right boundary ($\omega_{2\to 3} = \frac{M_*}{\sigma}$) it approaches $\rho_{2\to 3} = 2-2 = 0$.

V. Proof outline

The most natural basis for solving the weighted problem (3) is the Hermite polynomial basis, since the Hermite polynomials are orthonormal with respect to the weight $W(\omega)$. We therefore write the unknown $g(\omega)$ as a converging Hermite polynomial series (see Proposition 1)

$$g(\omega) = \sum_{n=0}^{\infty} c_n H_n(\omega), \tag{6}$$

where $H_n(\omega)$ is the normalized Hermite polynomial of degree n [14, Chapter 22], and $\{c_n\}$ are given by

$$c_n = \int_{\mathbb{R}} g(\omega) e^{-\omega^2} H_n(\omega) d\omega = \int_{\mathbb{R}} h(\omega) \varphi_n(\omega) d\omega,$$

where $\varphi_n(\omega) := W(\omega) H_n(\omega)$ is the orthonormal Hermite function of order n, and $h(\omega) = g(\omega) W(\omega)$. Then, the solution to (3) can be written as

$$\widetilde{g}(\omega) = \sum_{n \leqslant M} \widetilde{c_n} H_n(\omega), \tag{7}$$

where the coefficients $\underline{c_n} := \{c_n\}_{n \leqslant M}$ are given by

$$\underline{\widetilde{c_n}} = \arg\min_{c_n} \left\| \underline{y} - \sum_{n \leqslant M_{opt}} c_n \varphi_n(\underline{\omega}) \right\|_2. \tag{8}$$

Proposition 1. *If $g \in S_\sigma$, the series (6) converges, and furthermore we have the bound (excluding factors sub-exponential in n)*

$$|c_n| \sim \exp\left(-\frac{n}{2}\log\frac{2n}{e\sigma^2}\right) = \sigma^n \exp\left(-\frac{n}{2}\log\frac{2n}{e}\right). \tag{9}$$

Proof: The convergence follows from the general theory of Hermitian series [15], [16], as the function $h(\omega)$ has Gaussian decay on the real axis. In the special case $g = \exp(\sigma\omega)$, the c_n are given explicitly [17, Formula 7.374(6)] as

$$c_n = e^{\frac{\sigma^2}{4}} \frac{\sigma^n \pi^{\frac{1}{4}}}{\sqrt{2^n n!}}. \tag{10}$$

Using Stirling's approximation for the factorial function, the bound (9) follows. The claim for arbitrary $g \in S_\sigma$ is easily shown using Fubini's theorem and the representation (1). ■

Let $E = (M, N, \Omega)$ denote the parameters of the least squares problem (in the meantime we let M be arbitrary), and let V_E denote the Vandermonde matrix $V_E := [\varphi_n(\underline{\omega})]_{n \leqslant M}$. The least squares solution is given by ($V_E^\dagger$ denotes the Moore-Penrose pseudoinverse)

$$\underline{\widetilde{c_n}} = \left(V_E^T V_E\right)^{-1} V_E^T \underline{y} = V_E^\dagger \underline{y}.$$

As $|c_n|$ decay (super-) exponentially, it is not difficult to show (see [13] for details) that

$$\|\Delta \underline{c_n}\| := \|\underline{\widetilde{c_n}} - \underline{c_n}\| \sim \frac{\sqrt{N}}{\sigma_{min}(V_E)} (\epsilon + |c_M|).$$

Using the well-known estimates for the convergence of the trapezoidal rule (see e.g. [18, Table 6.1]) and the properties of the Hermite basis $\{\varphi_n\}$, it can be further shown that with the choice $\Delta\omega \sim N^{-1/2}$, $N = 2M$ and $\Omega = \sqrt{2M+1}$ the matrix $V_E^T V_E$ is diagonally dominant, and in fact its eigenvalues are exponentially close to 1. Therefore, disregarding terms which are polynomial in M and N, the corresponding error $B(\omega)$ can be bounded by

$$B(\omega) \leqslant (\epsilon + |c_M|) \sum_{n \leqslant M} |H_n(\omega)| + \sum_{n > M} |c_n H_n(\omega)|. \tag{11}$$

Our choice of M_* is made so that $|c_{M_*}| \sim \epsilon$. Using (9), this suggests

$$M_* = \frac{\log\frac{1}{\epsilon}}{q}, \quad q = q(\sigma, \epsilon) := \frac{1}{2}\mathcal{W}\left(\frac{4}{e\sigma^2}\log\frac{1}{\epsilon}\right), \tag{12}$$

where $\mathcal{W}$ is the Lambert's W-function.

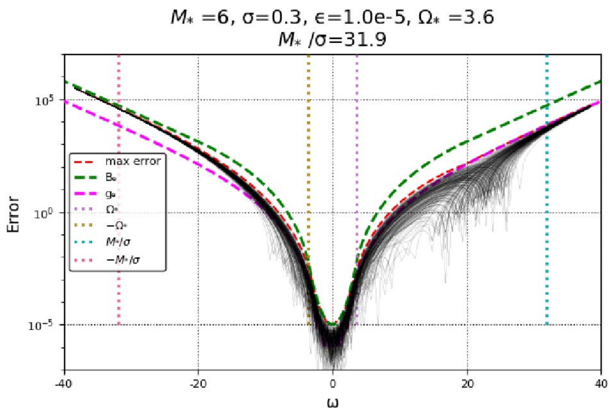
The asymptotics of Hermite polynomials are well-known, see e.g. the classical Szego's monograph [19] and also [20]. For instance, the following bounds hold uniformly in n for a fixed ω:

$$|H_n(\omega)| \asymp \begin{cases} \exp\left(\frac{\omega^2}{2}\right) & n > \frac{\omega^2 - 1}{2}, \\ \exp\left\{\frac{\omega^2}{2}(1-\alpha)\left(1 + \frac{1+\alpha}{2}\log\frac{1+\alpha}{1-\alpha}\right)\right\} & n < \frac{\omega^2-1}{2}, \end{cases}$$

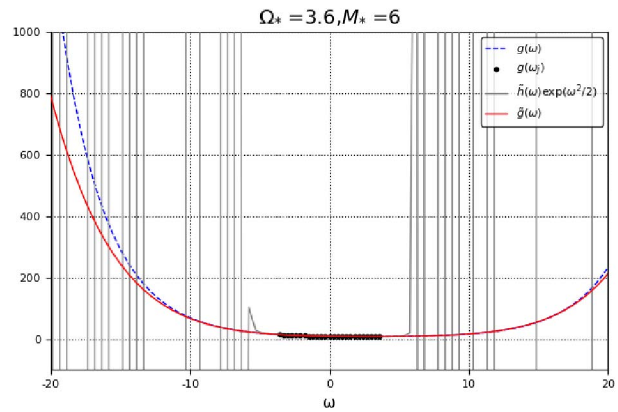
where $\alpha(n, \omega) = \sqrt{1 - \frac{2n+1}{\omega^2}}$. From these bounds it can be shown that for a fixed ω, the maximal term in the series $\{|c_n H_n(\omega)|\}_n$ is at $n_{max} \sim \sigma|\omega|$. This fact, together with (11) and (12) imply that for any $g \in S_\sigma$ we have

$$B_{\epsilon,\sigma}^{\tilde{g}}(\omega) \lessapprox \begin{cases} \epsilon \exp\left(\frac{\omega^2}{2}\right) & |\omega| \leqslant \Omega_*, \\ \epsilon |H_{M_*}(\omega)| & \Omega_* < |\omega| < \frac{M_*}{\sigma}, \\ |c_{\sigma|\omega|} H_{\sigma|\omega|}(\omega)| & |\omega| \geqslant \frac{M_*}{\sigma}. \end{cases}$$

Further simplifications give the first part of Theorem 1.

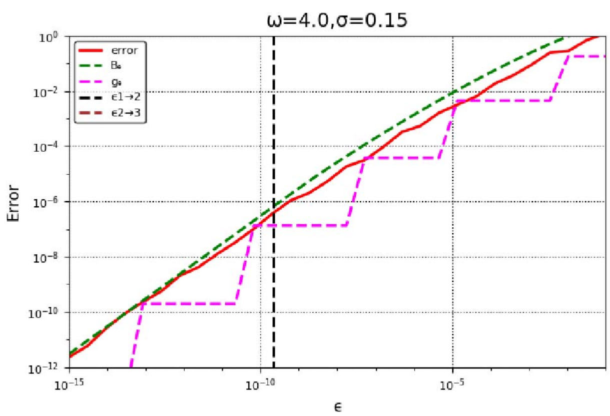


(a) The error and the bounds. Thin black lines are $B^{\tilde{g}}$ for different realizations of $\tilde{g}$, their maximal envelope is the red curve. The green curve is the analytical bound $B_{\sigma,\epsilon}^{opt}$, while the magenta curve is the minimax function $g_{\sigma,\epsilon}$. The dotted vertical lines are the region boundaries.

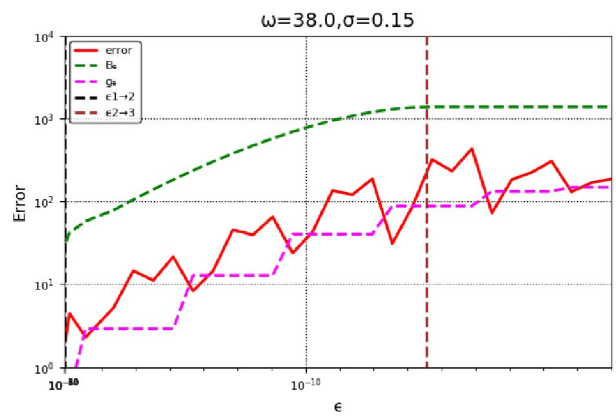


(b) The function (blue dashed) and its extrapolant (red solid). The black dots are the actual sampling points, while the grey curve is the noisy function $\frac{1}{W(\omega)}\tilde{h}(\omega)$ which would need to have been used without extrapolation.

Fig. 2: Results for $g(\omega) = 5 + \cosh(\sigma\omega - 2) + \sinh(\sigma\omega)$ with $\sigma = 0.3, \epsilon = 10^{-5}$.



(a) $\omega = 4$, the transition between regions 1 and 2 (black line)



(b) $\omega = 38$, the transition between regions 2 and 3 (brown line)

Fig. 3: Error vs bound for $\sigma = 0.15$, as function of ϵ. The same $g(\omega)$ as in Fig.2.

To show the optimality of the bound, we consider the following function (the coefficients $c_n(\sigma)$ are given in (10)):

$$g_{\sigma,\epsilon}(\omega) := \sum_{n \geqslant M_*} \frac{c_n(\sigma) + c_n(-\sigma)}{2} H_n(\omega)$$

$$= \cosh(\sigma\omega) - \sum_{n<M_*} e^{\frac{\sigma^2}{4}} \frac{\sigma^n \pi^{\frac{1}{4}}(1 + (-1)^n)}{\sqrt{2^n n!}} H_n(\omega).$$

Clearly, $g_{\sigma,\epsilon}(\omega) \in S_\sigma$. Furthermore, by essentially repeating the argument above it could be shown that $g_{\sigma,\epsilon}(\omega) \asymp B_{\sigma,\epsilon}^{opt}(\omega)$.

VI. Conclusions and future work

In this note we considered the problem of stable recovery of a bandlimited function f with bandlimit σ from its samples multiplied by a fast-decaying Gaussian window and further corrupted by an additive bounded noise. We have shown that a least-squares polynomial approximation with judiciously chosen degree M_* from equispaced samples (whose number scales linearly with M_*) provides an asymptotically optimal extension of the function outside of the essential support of the window, with the error which scales like a fractional power of the noise level ϵ, up to a length scale of $\sigma^{-1}\log\frac{1}{\epsilon}$. If the function f is a Fourier transform of a time/space limited signal $x(t)$, our results provide estimates on the possible resolution enhancement for x corrupted by a Gaussian convolution filter, while utilizing the a-priori information on its compact support.

One natural extension of this work would be to consider other fast-decaying windows. Another possible line of investigation is to generalize the time domain signal model to have a small number of shifted and compactly-supported "clumps", or approximate spikes (in the present note we considered a single such clump centered at the origin), converting the problem into a "blind superresolution" setting (see e.g. [21]); however in this case the reconstruction algorithm will have to be nonlinear.

Hölder-type pointwise stability has been observed in related works [13], [22]–[24], while in [12] it was shown that extrapolation for bandlimited functions (with a fixed bandlimit 2π, bounded on $\mathbb{R}$) is possible up to length scale of $\log \frac{1}{\epsilon}$. We plan to investigate connections to these works in a future publication.

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