MATHEMATICS OF COMPUTATION
Volume 84, Number 295, September 2015, Pages 2329–2350
S 0025-5718(2015)02948-2
Article electronically published on February 19, 2015

COMPLETE ALGEBRAIC RECONSTRUCTION OF PIECEWISE-SMOOTH FUNCTIONS FROM FOURIER DATA

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ABSTRACT. In this paper we provide a reconstruction algorithm for piecewise-smooth functions with a priori known smoothness and a number of discontinuities, from their Fourier coefficients, possessing the maximal possible asymptotic rate of convergence—including the positions of the discontinuities and the pointwise values of the function. This algorithm is a modification of our earlier method, which is in turn based on the algebraic method of K. Eckhoff proposed in the 1990s. The key ingredient of the new algorithm is to use a different set of Eckhoff's equations for reconstructing the location of each discontinuity. Instead of consecutive Fourier samples, we propose to use a "decimated" set which is evenly spread throughout the spectrum.

1. INTRODUCTION

Consider the problem of reconstructing a function $f : [-\pi, \pi] \to \mathbb{R}$ from a finite number of its Fourier coefficients

$$c_k(f) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, \mathrm{e}^{-\imath kt}\, \mathrm{d}\, t, \qquad k = 0, 1, \ldots, M.$$

It is well known that for periodic smooth functions, the truncated Fourier series

$$\mathfrak{F}_M(f) \stackrel{\text{def}}{=} \sum_{|k|=0}^{M} c_k(f)\, \mathrm{e}^{\imath kx}$$

converges to f very fast, subsequently making Fourier analysis attractive for many applications. The precise dependence of the rate of convergence on structural properties of f is extensively investigated in classical harmonic analysis and approximation theory (see e.g. [40]). In applications, it is often sufficient to consider the number of continuous derivatives of the function.

Definition 1. Let C^{d+1} denote the class of continuous functions having d continuous derivatives, such that in addition $f^{(d+1)}$ is piecewise-continuous and piecewise-differentiable.

Applying integration by parts and the Riemann-Lebesgue lemma one has immediately the following fact (see e.g. [24, Section 3]).

Received by the editor December 2, 2012 and, in revised form, November 30, 2013.
2010 *Mathematics Subject Classification.* Primary 65T40; Secondary 65D15.
Key words and phrases. Fourier inversion, nonlinear approximation, piecewise-smooth functions, Eckhoff's conjecture, Eckhoff's method, Gibbs phenomenon.
This research has been supported by the Adams Fellowship Program of the Israel Academy of Sciences and Humanities.

Proposition 2. *For any $f \in C^{d+1}$ which is periodic (including its first d derivatives), we have $|c_k(f)| = O\left(|k|^{-d-2}\right)$, while the approximation error is of the order*

$$|f(x) - \mathfrak{F}_M(f)(x)| = O\left(M^{-d-1}\right), \tag{1.1}$$

and this holds uniformly in $[-\pi, \pi]$.

Yet many realistic phenomena exhibit discontinuities, in which case the unknown function f is only piecewise-smooth. As a result, the trigonometric polynomial $\mathfrak{F}_M(f)$ no longer provides a good approximation to f due to the slow convergence of the Fourier series (one of the manifestations of this fact is commonly known as the "Gibbs phenomenon"). It has very serious implications, for example, when using spectral methods to calculate solutions of PDEs with shocks [24].

Definition 3. Let $PC(d+1, K)$ denote the class of piecewise-smooth functions f with K points of discontinuity of the first kind, such that the restriction of f on each continuity interval is in C^{d+1} (as in Definition 1).

An important question arises: *"Can such piecewise-smooth functions be reconstructed from their Fourier measurements, with accuracy which is comparable to the 'classical' one such as* (1.1)*"?*

This problem has received much attention, especially in the last few decades ([3–5, 11–14, 16, 18, 19, 21–23, 25, 26, 28–32, 35–38] would be only a partial list). It has long been known that the key problem for Fourier series acceleration is the detection of the shock locations. While efficient methods for edge detection exist (e.g. concentration kernels of Tadmor et al. [21, 22, 37]), the theoretical analysis of these methods suggests that they provide not more than first order accuracy. In contrast, our main interest in this paper is to investigate achievability of the maximal theoretically possible rate of convergence. Applying elementary considerations we have the following fact (see the proof in Appendix A).

Proposition 4. *Let $f \in PC(d+1, K)$. Then no deterministic algorithm can restore the locations of the discontinuities from $\{c_k(f)\}_{|k| \leqslant M}$ with accuracy which is asymptotically higher than M^{-d-2}.*

Until now, the question of whether this maximal accuracy is achievable remained open. During the 1990s, a certain method was put forward by K. Eckhoff in a series of papers [17–19], which he conjectured to provide such accuracy (see Section 2). Thus we have the following *"Eckhoff's conjecture"*.

Conjecture 5 (Eckhoff's conjecture)**.** *The jump locations of a piecewise-smooth function $f \in PC(d+1, K)$ can be reconstructed from its first $2M+1$ Fourier coefficients, with accuracy $O\left(M^{-d-2}\right)$, by solving the perturbed nonlinear system of algebraic equations* (2.3)*.*

In our previous work [9] we have provided an explicit reconstruction algorithm (Algorithm 1 on page 2332), based on original Eckhoff's method, which restored the jump locations (and subsequently the pointwise values of the function between the jumps) with "half" the maximal accuracy. In the present paper we modify the method of [9] (see Algorithm 2 and Algorithm 3) so that full asymptotic accuracy is achieved (Theorem 13). The vital difference of the new algorithm compared to the original Eckhoff's method (and its modification from [9]) is that when solving the system (2.3), instead of consecutive Fourier coefficients, we take ones that are evenly spaced throughout the whole sampling range (thus we call the new method "decimated Eckhoff's algorithm").

We describe the general approach, as well as our previous results obtained in [9], in Section 2. The modified algorithm is provided in Section 3, and its accuracy is analyzed in Section 4. Results of some numerical simulations are presented in Section 5. We briefly discuss the optimality and some practical aspects of the algebraic reconstruction algorithms in Section 6. Some possible extensions and generalizations are outlined in Section 7.

2. Eckhoff's method and half-order reconstruction

Let us first briefly describe what has become known as Eckhoff's method (or the Krylov-Gottlieb-Eckhoff method) for nonlinear Fourier reconstruction of piecewise-smooth functions [17–19].

Let $f \in PC(d+1, K)$. Consequently, f has $K > 0$ jump discontinuities $\{\xi_j\}_{j=1}^K$ (they can be located also at $\pm\pi$, but not necessarily so). Furthermore, in every segment $[\xi_{j-1}, \xi_j]$ we have that $f \in C^{d+1}$. Denote the associated jump magnitudes at ξ_j by

$$a_{\ell,j} \stackrel{\text{def}}{=} f^{(\ell)}(\xi_j^+) - f^{(\ell)}(\xi_j^-), \qquad \ell = 0, 1, \ldots, d.$$

We write the piecewise smooth f as the sum $f = \Psi + \Phi$, where $\Psi \in C^{d+1}$ and $\Phi(x)$ is a piecewise polynomial of degree d, uniquely determined by $\{\xi_j\}, \{a_{\ell,j}\}$ such that it "absorbs" all the discontinuities of f and its first d derivatives. This idea is very old and goes back at least to A.N. Krylov ([4, 27]). Eckhoff derives the following explicit representation for $\Phi(x)$:

$$\begin{aligned} \Phi(x) &= \sum_{j=1}^{K} \sum_{\ell=0}^{d} a_{\ell,j} V_\ell(x; \xi_j) \\ V_n(x; \xi_j) &= -\frac{(2\pi)^n}{(n+1)!} B_{n+1}\left(\frac{x-\xi_j}{2\pi}\right) \qquad \xi_j \le x \le \xi_j + 2\pi, \end{aligned} \tag{2.1}$$

where $V_n(x; \xi_j)$ is understood to be periodically extended to $[-\pi, \pi]$ and $B_n(x)$ is the n-th Bernoulli polynomial. Elementary integration by parts gives the following formula.

Proposition 6. *Let $\Phi(x)$ be given by (2.1). Then*

$$c_k(\Phi) = \frac{1}{2\pi} \sum_{j=1}^{K} \mathrm{e}^{-\imath k \xi_j} \sum_{\ell=0}^{d} (\imath k)^{-\ell-1} a_{\ell,j}. \tag{2.2}$$

Eckhoff observed that if Ψ is sufficiently smooth, then the contribution of $c_k(\Psi)$ to $c_k(f)$ is negligible **for large** k, and therefore one can hope to reconstruct the unknown parameters $\{\xi_j, a_{\ell,j}\}$ from the perturbed equations

$$c_k(f) = \frac{1}{2\pi} \sum_{j=1}^{K} \mathrm{e}^{-\imath k \xi_j} \sum_{\ell=0}^{d} (\imath k)^{-\ell-1} a_{\ell,j} + O\left(k^{-d-2}\right), \ k \gg 1. \tag{2.3}$$

His proposed method was to construct from the known values

$$\{c_k(f)\} \qquad k = M - (d+1)K + 1, M - (d+1)K + 2, \ldots, M$$

a system of algebraic equations satisfied by the jump points $\{\xi_1,\dots,\xi_K\}$, and solve this system numerically. Based on some explicit computations for small values of d,K and large number of numerical experiments, he conjectured that his method would reconstruct the jump locations with accuracy M^{-d-2} (Conjecture 5). In [9] we proposed a reconstruction method based on the original Eckhoff procedure, outlined in Algorithm 1.

Algorithm 1 Half-order algorithm, [9].

Let $f\in PC\left(d+1,K\right)$, and assume that $f=\Phi^{(d)}+\Psi$ where $\Phi^{(d)}$ is the piecewise polynomial absorbing all discontinuities of f, and $\Psi\in C^{d+1}$. Assume in addition the following a priori bounds:

- Minimal separation distance between the jumps
$$\min_{i\neq j}|\xi_i-\xi_j|\geqslant J>0.$$
- Upper bound on jump magnitudes
$$|a_{l,j}|\leqslant A<\infty.$$
- Lower bound on the value of the lowest-order jump
$$|a_{0,j}|\geqslant B>0.$$
- Upper bound on the size of the Fourier coefficients of Ψ:
$$|c_k\left(\Psi\right)|\leqslant R\cdot k^{-d-2}.$$

Let the first $M\gg 1$ Fourier coefficients of f be given for $M>M\left(d,K,J,A,B,R\right)$ (a quantity which is computable). The reconstruction is as follows.

(1) Obtain first-order approximations to the jump locations $\{\xi_1,\dots,\xi_K\}$ by Prony's method (Eckhoff's method of order 0).
(2) Localize each discontinuity ξ_j by calculating the first M Fourier coefficients of the function $f_j=f\cdot h_j$ where h_j is a C^∞ bump function satisfying
  (a) $h_j\equiv 0$ on the complement of $[\xi_j-J,\xi_j+J]$;
  (b) $h_j\equiv 1$ on $\left[\xi_j-\frac{J}{3},\xi_j+\frac{J}{3}\right]$.
(3) Fix the reconstruction order $d_1\leq\left\lfloor\frac{d}{2}\right\rfloor$. For each $j=1,2,\dots,K$, recover the parameters $\left\{\xi_j,a_{0,j},\dots,a_{d_1,j}\right\}$ from the approximate system of d_1+2 equations

$$c_k\left(f_j\right)=\frac{1}{2\pi}\,\mathrm{e}^{-\imath\xi_j k}\sum_{\ell=0}^{d_1}\frac{a_{\ell,j}}{\left(\imath k\right)^{\ell+1}}+\delta_k,\qquad k=M-d_1-1,M-d_1,\dots,M, \tag{2.4}$$

by Eckhoff's method for one jump. The actual method is to solve a single polynomial equation of degree d_1 constructed from the measurements $\{c_k\left(f_j\right)\}$, thus recovering the unknown ξ_j, and subsequently solve a linear system w.r.t. the rest of the parameters $\left\{a_{0,j},\dots,a_{d_1,j}\right\}$.
(4) From the previous steps we obtained approximate values for the parameters $\left\{\widetilde{\xi_j}\right\}$ and $\{\widetilde{a}_{\ell,j}\}$. The final approximation is taken to be

(2.5)
$$\widetilde{f}=\widetilde{\Psi}+\widetilde{\Phi}=\sum_{|k|\leq M}\left\{c_k(f)-\frac{1}{2\pi}\sum_{j=1}^{K}\mathrm{e}^{-\imath\widetilde{\xi_j}k}\sum_{\ell=0}^{d_1}\frac{\widetilde{a}_{\ell,j}}{(\imath k)^{\ell+1}}\right\}\mathrm{e}^{\imath kx}+\sum_{j=1}^{K}\sum_{\ell=0}^{d_1}\widetilde{a}_{\ell,j}V_\ell(x;\widetilde{\xi_j}).$$

We have also shown that this method achieves the following accuracy.

Theorem 7 ([9]). *Let $f \in PC(d+1, K)$ and let $\widetilde{f}$ be the approximation of order $d_1 \leqslant \left\lfloor \frac{d}{2} \right\rfloor$ computed by Algorithm 1. Then for large enough M we have*[1]

$$\begin{aligned} \left|\widetilde{\xi_j} - \xi_j\right| &\leqslant C_1(d, d_1, K, J, A, B, R) \cdot M^{-d_1-2}, \\ |\widetilde{a}_{\ell,j} - a_{\ell,j}| &\leqslant C_2(d, d_1, K, J, A, B, R) \cdot M^{\ell-d_1-1}, \ \ell = 0, 1, \ldots, d_1, \\ \left|\widetilde{f}(x) - f(x)\right| &\leqslant C_3(d, d_1, K, J, A, B, R) \cdot M^{-d_1-1}. \end{aligned} \tag{2.6}$$

The nontrivial part of the proof of this result was to analyze in detail the polynomial equation $p(\xi_j) = 0$ in step 3 of Algorithm 1. It turned out that additional orders of smoothness (namely, between d_1 and d) produced an error term δ_k in (2.4) which, when substituted into the polynomial p, resulted in unexpected cancellations due to which the root ξ_j was perturbed only by $O\left(M^{-d_1-2}\right)$. This phenomenon was first noticed by Eckhoff himself in [18] for $d = 1$, but at the time its full significance was not realized.

3. The decimated Eckhoff algorithm

In this section we present the "decimated Eckhoff algorithm", which has a single essential difference compared to Algorithm 1. The difference is that in step 3, we solve the "full-order" system, while choosing the indices k to be evenly distributed across the range $\{0, 1, \ldots, M\}$ (instead of the original choice $k = M - d - 1, M - d, \ldots, M$). That is, denoting

$$N \stackrel{\text{def}}{=} \left\lfloor \frac{M}{(d+2)} \right\rfloor,$$

the modified system (2.4) reads

$$\widetilde{c_k} = \underbrace{\frac{\omega^k}{2\pi} \sum_{\ell=0}^{d} \frac{a_\ell}{(\imath k)^{\ell+1}}}_{\stackrel{\text{def}}{=} c_k} + \epsilon_k, \quad k = N, 2N, \ldots, (d+2)N, \quad |\epsilon_k| \leq R \cdot k^{-d-2}. \tag{3.1}$$

Here $\omega = \mathrm{e}^{-\imath\xi}$ with $\xi = \xi_j \in [-\pi, \pi]$ being the unknown location of the (single) discontinuity of the localized function f_j (see step 2 of Algorithm 1).

The decimated system (3.1) is solved in two steps. First, a polynomial equation $q_N^d(u) = 0$ is constructed from the values $\{\widetilde{c_k}\}_{k=N,2N,\ldots,(d+2)N}$. This q_N^d is in fact a perturbation of an "exact" equation $p_N^d(u) = 0$, constructed from the unperturbed (and unknown) values $\{c_k\}$ as in (3.1). This p_N^d is defined explicitly below in (3.5). As we show in Proposition 9, one of the roots of this exact equation is the value $z = \omega^N$. Thus, by solving the perturbed equation $q_N^d(u) = 0$ we recover the unknown $\widetilde{z} = \mathrm{e}^{-\imath\widetilde{\xi}N}$, and by extracting the N-th root and subsequently taking logarithm we obtain the approximation to the jump $\widetilde{\xi}$. The operation of taking

[1] The last (pointwise) bound holds on "jump-free" regions.

root generally results in a multi-valued solution[2]. Therefore, to ensure correct reconstruction, we need an additional assumption that the jump ξ must be known with *a priori* accuracy of the order $o\left(N^{-1}\right)$. Once the approximate jump location $\widetilde{\xi}$ is reconstructed, the jump magnitudes $\{a_{\ell,j}\}_{\ell=0}^{d}$ are recovered by solving a linear system of equations (3.6).

The above procedure for recovery of a single jump is summarized in Algorithm 2 on the next page. The complete algorithm is outlined in Algorithm 3 on page 2336.

Let us now define the "exact" equation $p_N^d(u)=0$. Denote $\alpha_\ell = \imath^{d+1-\ell} a_{d-\ell}$ and let

$$m_k \stackrel{\text{def}}{=} \omega^k \sum_{\ell=0}^{d} \alpha_\ell k^\ell. \tag{3.2}$$

With this notation, multiply both sides of (3.1) by $(2\pi)(\imath k)^{d+1}$ and get

$$\widetilde{m}_k \stackrel{\text{def}}{=} 2\pi (\imath k)^{d+1} \widetilde{c_k} = m_k + \delta_k, \qquad k = N, 2N, \dots, (d+2)N, \quad |\delta_k| \leqslant R \cdot k^{-1}. \tag{3.3}$$

Recall that we have defined $z=\omega^N$. Therefore we have by (3.2)

$$m_{(j+1)N} = z^{j+1} \sum_{\ell=0}^{d} \alpha_\ell (j+1)^\ell N^\ell. \tag{3.4}$$

Definition 8. Let

$$p_N^d(u) \stackrel{\text{def}}{=} \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} m_{(j+1)N} u^{d+1-j}. \tag{3.5}$$

Proposition 9. *The point $u=z$ is a root of $p_N^d(u)$.*

Proof. From (3.4) and (3.5) we have

$$\begin{aligned} p_N^d(z) &= \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} z^{j+1} \sum_{\ell=0}^{d} \alpha_\ell (j+1)^\ell N^\ell z^{d+1-j} \\ &= z^{d+2} \sum_{\ell=0}^{d} \alpha_\ell N^\ell \left\{ \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} (j+1)^\ell \right\}. \end{aligned}$$

The expression in the curly braces is just the $d+1$-st forward difference operator applied to the polynomial function $\varphi(k)=k^\ell$. Since $\ell < d+1$, this is always zero (see e.g. [20]). □

Now let us explicitly write the linear system for the jump magnitudes.

[2]For example, if $N=2$, then the solution $z=1$ corresponds to either $\xi=0$ or $\xi=\pm\pi$. In the general case, there are N possible solutions, as follows:

$$\begin{aligned} e^{\imath \xi N} &= e^{\imath t}, \\ \xi N - t &= 2\pi n, \\ \xi &= \frac{t}{N} + \frac{2\pi}{N} n, \qquad n \in \mathbb{Z}. \end{aligned}$$

Definition 10. Let V_N^d denote the $(d+1)\times(d+1)$ matrix

$$V_N^d \stackrel{\text{def}}{=} \begin{bmatrix} 1 & N & N^2 & \dots & N^d \\ 1 & 2N & (2N)^2 & \dots & (2N)^d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & (d+1)N & ((d+1)N)^2 & \dots & ((d+1)N)^d \end{bmatrix}.$$

Note that V_N^d is the Vandermonde matrix on the points $\{N, 2N, \dots, (d+1)N\}$ and thus it is nondegenerate for all $N \geqslant 1$.

Proposition 11. *The vector of exact magnitudes* $\{\alpha_j\}$ *satisfies*

$$\begin{bmatrix} m_N \omega^{-N} \\ m_{2N}\omega^{-2N} \\ \vdots \\ m_{(d+1)N}\omega^{-(d+1)N} \end{bmatrix} = V_N^d \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_d \end{bmatrix}. \tag{3.6}$$

Proof. Immediately follows from (3.2). □

Algorithm 2 Recovery of single jump parameters

Let the first $N(d+2) \gg 1$ Fourier coefficients of the function f_j be given as in (3.1), and assume that the jump position ξ is already known with accuracy $o\left(N^{-1}\right)$.

(1) Construct the polynomial

$$q_N^d(u) \quad = \quad \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} \widetilde{m}_{(j+1)N} u^{d+1-j}$$

from the given perturbed measurements $\widetilde{m}_N, \widetilde{m}_{2N}, \dots, \widetilde{m}_{(d+2)N}$ as in (3.3).

(2) Find the root $\widetilde{z}$ which is closest to the unit circle (in fact any root will suffice, see Remark 19 below).

(3) Take $\widetilde{\omega} = \sqrt[N]{\widetilde{z}}$. Note that in general there are N possible values on the unit circle, but since we already know the approximate location of ω, the correct value can be chosen consistently.

(4) Set $\widetilde{\xi} = -\arg \widetilde{\omega}$.

(5) To recover the magnitudes, solve the perturbed linear system (3.6):

$$\begin{bmatrix} \widetilde{m}_N \widetilde{\omega}_N^{-N} \\ \widetilde{m}_{2N}\widetilde{\omega}_N^{-2N} \\ \vdots \\ \widetilde{m}_{(d+1)N}\widetilde{\omega}_N^{-(d+1)N} \end{bmatrix} = V_N^d \begin{bmatrix} \widetilde{\alpha}_0 \\ \widetilde{\alpha}_1 \\ \vdots \\ \widetilde{\alpha}_d \end{bmatrix}. \tag{3.7}$$

Algorithm 3 Full accuracy Fourier approximation

Let $f \in PC(d+1, K)$, and assume that $f = \Phi^{(d)} + \Psi$ where $\Phi^{(d)}$ is the piecewise polynomial absorbing all discontinuities of f, and $\Psi \in C^{d+1}$. Assume the a priori bounds as in Algorithm 1.

(1) Using Algorithm 1, obtain approximate values of the jumps (up to accuracy $O\left(N^{-\lfloor \frac{d}{2} \rfloor - 2}\right)$) and the Fourier coefficients of the functions f_j. (By Theorem 7 this is indeed possible.)
(2) Use Algorithm 2 to further improve the accuracy of reconstructing the jumps $\left\{\widetilde{\xi}_j\right\}_{j=1}^{K}$ and the magnitudes $\{\widetilde{a}_{\ell,j}\}$.
(3) Take the final approximation as defined in (2.5).

4. Main result

The key result of this paper is the following.

Theorem 12. *Assume that*

$$|a_\ell| \leqslant A^* < \infty, \quad |a_0| \geqslant B^* > 0. \tag{4.1}$$

Then for $N \gg 1$, Algorithm 2 recovers the parameters of a single jump from the data $\widetilde{m}_k$ (given by (3.3)) with the following accuracy:

$$\begin{aligned} \left|\widetilde{\xi} - \xi\right| &\leqslant C_4 \frac{R^*}{B^*} N^{-d-2}, \\ |\widetilde{\alpha}_\ell - \alpha_\ell| &\leqslant C_{5,\ell} R^* \left(1 + \frac{A^*}{B^*}\right) N^{-\ell-1}, \qquad \ell = 0, 1, \dots, d, \end{aligned}$$

where R^* is some constant for which (3.3) holds, C_4 depends only on d and $C_{5,\ell}$ depends only on ℓ and d.

An immediate consequence is the resolution of Eckhoff's conjecture.

Theorem 13. *Let $f \in PC(d+1, K)$ and let $\widetilde{f}$ be the approximation of order d computed by Algorithm 3. Then for $M \gg 1$,*

$$\begin{aligned} \left|\widetilde{\xi}_j - \xi_j\right| &\leqslant C_6(d, K, J, A, B, R) \cdot M^{-d-2}, \\ |\widetilde{a}_{\ell,j} - a_{\ell,j}| &\leqslant C_7(\ell, d, K, J, A, B, R) \cdot M^{\ell-d-1}, \qquad \ell = 0, 1, \dots, d, \\ \left|\widetilde{f}(x) - f(x)\right| &\leqslant C_8(d, K, J, A, B, R) \cdot M^{-d-1}. \end{aligned} \tag{4.2}$$

Proof. By Theorem 5.2 of [9], the Fourier coefficients of the localized functions f_j have error bounded by $R' k^{-d-2}$ where the constant R' depends in general on all the a priori bounds, but not on M. Therefore the a priori bounds required by Theorem 12 are satisfied by $R^* = R'$, $A^* = A$ and $B^* = B$. Therefore, the estimates of Theorem 12 hold for each discontinuity $j = 1, \dots, K$. After substituting $M = (d+2)N$ and $\alpha_\ell = \imath^{d+1-\ell} a_{d-\ell,j}$, we get the first two lines of (4.2). To get the pointwise estimate $\left|\widetilde{f}(x) - f(x)\right|$, just repeat the proof of Theorem 6.1 of [9] *verbatim.* □

The remainder of this section is devoted to proving Theorem 12.

Let us first define an auxiliary polynomial sequence.

Definition 14. For all i,d nonnegative integers let

$$s_i^d(w) \stackrel{\mathrm{def}}{=} \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} (j+1)^i w^{d+1-j}.$$

Proposition 15. *Let $w = \frac{u}{z}$ (recall that $z = \omega^N$). Then*

$$p_N^d(u) = z^{d+2} \sum_{i=0}^{d} \alpha_i N^i s_i^d(w). \tag{4.3}$$

Proof. By (3.4) and (3.5) we have

$$\begin{aligned} p_N^d(zw) &= \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} z^{j+1} \sum_{i=0}^{d} \alpha_i (j+1)^i N^i (zw)^{d+1-j} \\ &= z^{d+2} \sum_{i=0}^{d} \alpha_i N^i \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} (j+1)^i w^{d+1-j} \\ &= z^{d+2} \sum_{i=0}^{d} \alpha_i N^i s_i^d(w). \end{aligned}$$

□

The most immediate conclusion of the formula (4.3) is that the asymptotic properties of the polynomials p_N^d are eventually determined by the corresponding properties of the fixed polynomial s_d^d.

Lemma 16. *The polynomial $s_d^d(w)$ is square-free, and all of its roots belong to the interval* $[1, +\infty)$.

Proof. We divide the proof into several steps.

(1) First, notice that we have the following recursion:

$$s_{i+1}^d(w) = (d+2) s_i^d(w) - w \frac{\mathrm{d}}{\mathrm{d}\,w} s_i^d(w). \tag{4.4}$$

Indeed,

$$\begin{aligned} (d+2) s_i^d(w) - w \frac{\mathrm{d}}{\mathrm{d}\,w} s_i^d(w) &= (d+2) \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} (j+1)^i w^{d+1-j} \\ &\quad - w \sum_{j=0}^{d} (-1)^j (d+1-j) \binom{d+1}{j} (j+1)^i w^{d+1-j} \\ &= \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} (j+1)^i w^{d+1-j} \{d+2-(d+1-j)\} \\ &= \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} (j+1)^{i+1} w^{d+1-j} \\ &= s_{i+1}^d(w). \end{aligned}$$

(2) Next, notice that

$$s_{i+1}^d(w) = w^{d+3} \frac{\mathrm{d}}{\mathrm{d}\,w} \left[\frac{1}{w^{d+2}} s_i^d(w) \right]. \tag{4.5}$$

(3) By Rolle's theorem applied to (4.5), we obtain that there is a root of s_{i+1}^d between any two consecutive roots of s_i^d.
(4) Direct computation gives, for $i=1$,

$$s_1^d(w) = (w-1)^d (w-(d+2)),$$

and therefore the biggest root of $s_1^d(w)$ is simple. Let us show by induction that this property is preserved for all $i \geqslant 1$. Let $y_i > 0$ be the biggest root of s_i^d, which is by assumption simple. Since the leading coefficient of s_i^d is positive, we must have that $\frac{\mathrm{d}}{\mathrm{d}\,w} s_i^d(w)\big|_{w=y_i} > 0$. Therefore, by (4.4) we get $s_{i+1}^d(y_i) < 0$, so there must be a root of s_{i+1}^d bigger than y_i. By counting roots and using item (3), this new root must be simple.
(5) Starting with $s_0^d(w) = (w-1)^{d+1}$, all the above implies that s_i^d has exactly $d+1$ real roots, among them $w=1$ with multiplicity $d+1-i$ and all the rest of the roots being simple and bigger than 1.

The proof is finished by considering the last item for $i=d$. □

Recall Proposition 9. Let $\left\{u_1^{(N)} = z, \ldots, u_d^{(N)}\right\}$ denote the roots of $p_N^d(u)$, and $\{w_1 = 1, \ldots, w_d\}$ denote the roots of $s_d^d(w)$.

Proposition 17. *The pairwise distances between* $\left\{u_1^{(N)}, \ldots, u_d^{(N)}\right\}$ *remain* $O(1)$ *as* $N \to \infty$.

Proof. Consider the decomposition (4.3). By Lemma 16, $\{w_1, \ldots, w_d\}$ are positive, real and simple roots of $s_d^d(w)$. By Rouche's theorem, as $N \to \infty$ the roots of $\frac{1}{N^d} p_N^d(zw)$ converge to $\{w_1, \ldots, w_d\}$. Obviously the polynomials $\frac{1}{N^d} p_N^d(zw)$ and $p_N^d(zw)$ have the same roots, therefore $\left\{u_1^{(N)}, \ldots, u_d^{(N)}\right\}$ also converge to $\{w_1, \ldots, w_d\}$. Since the pairwise distances between the fixed numbers $\{w_1, \ldots, w_d\}$ do not depend on N, this finishes the proof. □

Now we can estimate the deviation of the roots of q_N^d from $\left\{u_1^{(N)}, \ldots, u_d^{(N)}\right\}$.

Lemma 18. *Denote by* $\left\{y_1^{(N)}, \ldots, y_d^{(N)}\right\}$ *the roots of* q_N^d, *and assume the a priori bounds of Theorem* 12. *Then there exists* $C_9 = C_9(d)$ *such that for* $N \gg 1$ *and for* $j = 1, 2, \ldots, d$

$$\left| y_j^{(N)} - u_j^{(N)} \right| \leqslant C_9 \frac{R^*}{B^*} N^{-d-1}.$$

Proof. The proof is based on the application of Rouche's theorem. Using the decomposition (4.3) and Lemma 16, we have that, for $N \gg 1$,

$$\left| \frac{\mathrm{d}}{\mathrm{d}\,u} p_N^d(u)\big|_{u=u_i^{(N)}} \right| \approx |\alpha_d| N^d, \qquad i = 1, 2, \ldots, d. \tag{4.6}$$

In particular, this means that there exists a constant $C_{10} = C_{10}(d)$ such that for all $i = 1, 2, \ldots, d$ and $N \gg 1$,

$$\left| \frac{\mathrm{d}}{\mathrm{d}\,u} p_N^d(u)\big|_{u=u_i^{(N)}} \right| \geqslant C_{10} B^* N^d. \tag{4.7}$$

Again, from (4.3) it is easy to see that for $N \gg 1$, the high-order derivatives of p_N^d at $u_i^{(N)}$ can be uniformly bounded by an estimate of the form

$$\left| \frac{\mathrm{d}^k}{\mathrm{d}\, u^k} p_N^d (u) \big|_{u=u_i^{(N)}} \right| \leqslant C_{11} A^* N^d, \quad k = 2, \ldots, d$$

for some constant $C_{11} = C_{11}(d)$.

Next we take disks of radius $\eta(N) = C_9 \frac{R^*}{B^*} N^{-d-1}$ around each root $u_i^{(N)}$, where C_9 is to be determined. Let us fix $1 \leqslant i \leqslant d$, and consider the circles

$$\gamma_i^{(N)} = \left\{ t_\phi = u_i^{(N)} + \eta(N) \mathrm{e}^{\imath\phi},\ 0 \leqslant \phi < 2\pi \right\}.$$

By the Taylor formula we have for each $t_\phi \in \gamma_i^{(N)}$,

$$\begin{aligned}
\left| p_N^d (t_\phi) \right| &= \Bigg| \underbrace{p_N^d \left(u_i^{(N)} \right)}_{=0} + \underbrace{\frac{\mathrm{d}}{\mathrm{d}\, u} p_N^d \left(u_i^{(N)} \right)}_{|\cdot| \geqslant C_{10} B^* N^d,\ (4.7)} \eta(N) \mathrm{e}^{\imath\phi} \\
&\qquad + \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}\, u^2} p_N^d \left(u_i^{(N)} \right) \eta^2(N) \mathrm{e}^{2\imath\phi} + \ldots \Bigg| \\
{}_{(N \gg 1)} &\geqslant C_{12} B^* \eta(N) N^d = C_{12} C_9 R^* N^{-1} \qquad (C_{12} = 2C_{10}).
\end{aligned}$$

Now consider the perturbation polynomial $e_N^d \overset{\mathrm{def}}{=} q_N^d - p_N^d$. Its coefficients have magnitudes $\left| \widetilde{m_{jN}} - m_{jN} \right| \leqslant R^* N^{-1}$. Therefore

$$\left| e_N^d (t_\phi) \right| \leqslant C_{13} R^* N^{-1}.$$

Note that for $N \gg 1$ the constant C_{13} does not depend on C_9 because, say, $|t_\phi| \leqslant 2 \left| u_i^{(N)} \right| < C^\sharp (d)$, an absolute constant.

Consequently, if we choose $C_9 = 2\frac{C_{13}}{C_{12}} > \frac{C_{13}}{C_{12}}$ we can apply Rouche's theorem and conclude that q_N^d has a simple zero within distance $C_9 \frac{R^*}{B^*} N^{-d-1}$ from $u_i^{(N)}$.

By Proposition 17 the $\left\{ u_i^{(N)} \right\}$ are $O(1)$-separated, therefore if N is large enough then the quantity $C_9 \frac{R^*}{B^*} N^{-d-1}$ will be smaller than the minimal separation distance. □

Remark 19. This analysis is valid for *any root* of q_N^d, not just the perturbation of $u_1 = z$. The roots of q_N^d all lie approximately on the ray with angle ξN. This means that the parameter ξ can be recovered with high accuracy from any root of $q_N^d(u)$, and we expect that it might be important for practice (so for instance one can approximate ξ by averaging).

Proof of Theorem 12, *first part.* Let us track steps 2–4 of Algorithm 2.

- By Lemma 18, the accuracy of step 2 is bounded by $C_9 \frac{R^*}{B^*} N^{-d-1}$, i.e., we can write
$$\widetilde{z}_N = z + \frac{C^*(N)}{N^{d+1}},$$
where $|C^*(N)| \leqslant C_9 \frac{R^*}{B^*}$.

- Extraction of N-th root in step 3 further decreases the error by the factor $\frac{1}{N}$. Indeed, we have

$$\begin{aligned}
\left|\widetilde{\omega}_N-\omega\right| &= \left|1-\frac{\widetilde{\omega}_N}{\omega}\right|=\left|1-\left(\frac{\widetilde{z}_N}{z}\right)^{\frac{1}{N}}\right| \\
{\scriptstyle\left(|C^{**}(N)|\leqslant C_9\frac{R^*}{B^*}\right)} &= \left|1-\left(1+\frac{C^{**}(N)}{N^{d+1}}\right)^{\frac{1}{N}}\right| \\
{\scriptstyle(\text{Bernoulli's inequality})} &\leqslant C_9\frac{R^*}{B^*}N^{-d-2}.
\end{aligned}$$

- Step 4 preserves this estimate, since

$$\begin{aligned}
\widetilde{\omega}_N &= \omega+C_{\$}(N)\frac{R^*}{B^*}N^{-d-2}, \qquad |C_{\$}(N)|\leqslant C_9 \\
\implies \left|\widetilde{\xi}_N-\xi\right| &= \left|\log\frac{\widetilde{\omega}_N}{\omega}\right|=\left|\log\left(1+C_{\$\$}(N)\frac{R^*}{B^*}N^{-d-2}\right)\right| \qquad (|C_{\$\$}(N)|\leqslant C_9) \\
&\leqslant 2C_9\frac{R^*}{B^*}N^{-d-2},
\end{aligned}$$

the last inequality following from the estimate $|\log(1+\varepsilon)|<2|\varepsilon|$ for $|\varepsilon|\ll 1$.

The proof of the first part is therefore finished with $C_4 \overset{\text{def}}{=} 2C_9$. □

Proof of Theorem 12, *second part.* We have recovered the approximate value $\widetilde{\omega}_N$ which satisfies $|\widetilde{\omega}_N-\omega|\leq C_4\frac{R^*}{B^*}N^{-d-2}$, while $|\widetilde{m}_k-m_k|\leq R^*k^{-1}$. Now we estimate the corresponding error in the solution of the linear system (3.7).

By (3.6) and (3.7), the error vector satisfies

$$\begin{bmatrix}\widetilde{\alpha}_0-\alpha_0\\ \widetilde{\alpha}_1-\alpha_1\\ \vdots\\ \widetilde{\alpha}_d-\alpha_d\end{bmatrix}=\left(V_N^d\right)^{-1}\begin{bmatrix}\widetilde{m}_N\widetilde{\omega}_N^{-N}-m_N\omega^{-N}\\ \widetilde{m}_{2N}\widetilde{\omega}_N^{-2N}-m_{2N}\omega^{-2N}\\ \vdots\\ \widetilde{m}_{(d+1)N}\widetilde{\omega}_N^{-(d+1)N}-m_{(d+1)N}\omega^{-(d+1)N}\end{bmatrix}.$$

Since

$$\begin{aligned}
|m_{jN}| &\leq C_{14}A^*N^d, \\
\widetilde{m_{jN}} &= m_{jN}+R(N)N^{-1}, \qquad R(N)\leqslant R^*, \\
\widetilde{\omega}_N &= \omega+C_{\$}(N)\frac{R^*}{B^*}N^{-d-2}, \qquad C_{\$}(N)\leqslant C_4,
\end{aligned}$$

then we have (using standard Taylor majorization techniques, see e.g. [9, Proposition A.7]) that

$$\begin{aligned}
\widetilde{\omega}_N^{-jN} &= \left(\omega+C_4(N)\frac{R^*}{B^*}N^{-d-2}\right)^{-jN} \\
&= \omega^{-jN}\left(1+\frac{C_4(N)R^*}{B^*\omega}N^{-d-2}\right)^{-jN} \\
&\quad \omega^{-jN}\left(1-C_{15}(N)\frac{R^*}{B^*}N^{-d-1}\right)
\end{aligned}$$

with $|C_{15}(N)| \leqslant 2C_9$ and, consequently,

$$
\begin{aligned}
&\left|\widetilde{m}_{jN}\widetilde{\omega}_N^{-jN} - m_{jN}\omega^{-jN}\right| \\
&= \left|\left(m_{jN} + R(N)\,N^{-1}\right)\omega^{-jN}\left(1 - C_{15}(N)\frac{R^*}{B^*}N^{-d-1}\right) - m_{jN}\omega^{-jN}\right| \\
&\leqslant \frac{R^*}{N}\left|\frac{2C_9C_{14}A^*}{B^*} + 1\right| + O\left(N^{-d-2}\right) \\
&\leq C_{16}R\left(1 + \frac{A^*}{B^*}\right)N^{-1}, \qquad C_{16} = \max\{1, 2C_9C_{14}\}.
\end{aligned}
\tag{4.8}
$$

Denote $\zeta_j = \widetilde{m}_{jN}\widetilde{\omega}_N^{-jN} - m_{jN}\omega^{-jN}$ and also let $C_{17,\ell}$ be an upper bound on the sum of absolute values of the entries in the ℓ-th row of $\left(V_1^d\right)^{-1}$. It is immediate that

$$
V_N^d = V_1^d \operatorname{diag}\left\{1, N, \dots, N^d\right\};
$$

therefore,

$$
\begin{bmatrix} \widetilde{\alpha}_0 - \alpha_0 \\ \widetilde{\alpha}_1 - \alpha_1 \\ \vdots \\ \widetilde{\alpha}_d - \alpha_d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & N^{-1} & & \\ & & \ddots & \\ & & & N^{-d} \end{bmatrix} \left(V_1^d\right)^{-1} \begin{bmatrix} \zeta_0 \\ \zeta_1 \\ \vdots \\ \zeta_d \end{bmatrix}.
$$

Using the estimate (4.8) we have immediately that, for $\ell = 0, 1, \dots, d$,

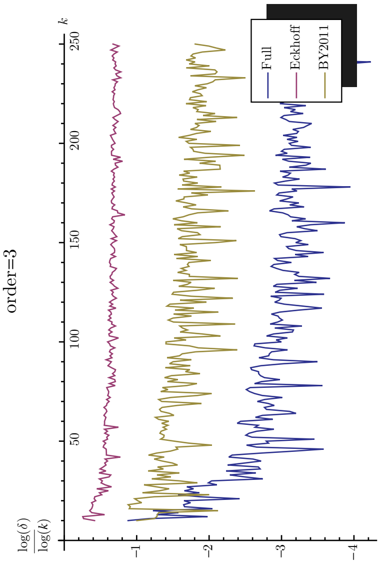
$$
|\widetilde{\alpha}_\ell - \alpha_\ell| \leqslant C_{5,\ell}R^*\left(1 + \frac{A^*}{B^*}\right)N^{-\ell-1},
$$

where $C_{5,\ell} \stackrel{\text{def}}{=} C_{16}C_{17,\ell}$. This completes the proof of Theorem 12. □

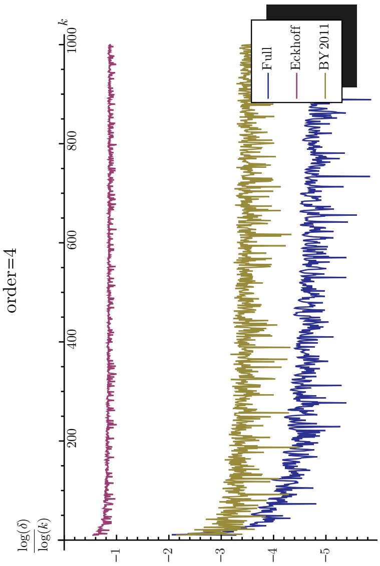
5. Numerical experiments

In our numerical experiments we compared the performance of the following Eckhoff-based methods for recovery of a single jump point position from the first M Fourier coefficients: original Eckhoff's formulation from [18] (Eckhoff); our previous method from [9] (BY 2011); the method presented in this paper (Full). All three methods in essense solve a polynomial equation $p_M(u) = 0$ satisfied by the jump point $\omega = \mathrm{e}^{-\imath\xi}$: for Eckhoff and BY 2011 this polynomial is constructed from consecutive samples $k = M - d - 1, \dots, M$ (Algorithm 1), while Full uses the decimated sequence $k = N, 2N, \dots, (d+2)N$. The only difference between Eckhoff and BY 2011 is the degree of $p_M(u)$: the former uses the full smoothness d while the latter uses $d_1 = \left\lfloor \frac{d}{2} \right\rfloor$. The jump point ξ and the magnitudes, as well as the error terms, are randomly chosen at the beginning of the whole experiment.

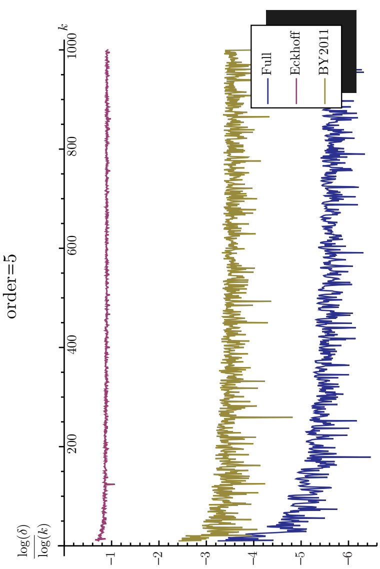
All calculations were done using Mathematica software with a high-precision setting. The results, presented in Figure 1, agree well with the theory: Full presents an improvement of $\sim M^{-d/2}$ compared to BY 2011, and improvement of order M^{-d} compared to Eckhoff.



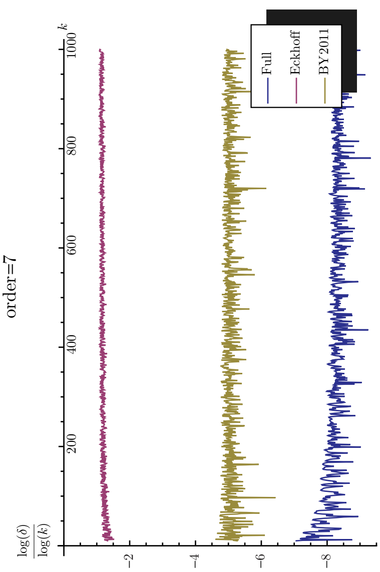
(A) Order=3



(B) Order=4



(C) Order=5



(D) Order=7

FIGURE 1. Full represents the algorithm of this paper, BY 2011 refers to the method of [9] while ECKHOFF denotes the original method of Eckhoff from [18]. The x axis shows the index k used for the reconstruction, corresponding to the number M in the text. The y axis shows the ratio $\frac{\log \delta}{\log k}$, where δ is the reconstruction error exhibited by the algorithms.

6. Practical aspects of algebraic reconstruction

6.1. **Stability of the algorithm and Prony-type systems.** The optimality, or efficiency, of the proposed algorithm remains an important practical issue. It is immediately seen that our algorithm is stable with respect to perturbations in the Fourier coefficients $c_k(f)$ *of the order* $O\left(k^{-d-2}\right)$ (since such perturbations will just be absorbed into the constant R appearing in (3.1)). This means, however, that the higher coefficients need to be acquired with increasing accuracy, which might very well be impossible in practice. While best possible asymptotic rate of convergence is achieved, it comes at the cost of high-precision computations and a large number of required Fourier coefficients (see e.g. experiments on localization procedure in [9] where convergence starts with large M). So in terms of *actual performance,* the "decimated Eckhoff algorithm" is probably not the best currently available method for jump detection in real-world scenarios. For this reason, at this stage we do not attempt to compare its performance to well-known methods such as concentration kernels. Instead, in this section we briefly discuss the question of best *absolute* performance of *any method whatsoever.*

Consider the Eckhoff's problem without reference to any concrete method. A formulation which might be more suitable for practical applications is the following.

Problem 20. Given first M Fourier coefficients of $f \in PC(d+1, K)$, possibly with some perturbations bounded by $\leqslant \delta$, find the points of discontinuity of f with *smallest absolute error.*

The problem is that, as far as we are aware, even the question of determining *what the smallest absolute error actually is*, remains open. Motivated by this question, we have started investigating the so-called "Prony type" systems[3] (of which (2.3) is a special case), in particular, lower bounds for their solution. Let us now briefly discuss the relevant results of [6, 10] in the context of Eckhoff's problem.

Consider the following "polynomial Prony" system of equations:

$$m_k = \sum_{j=1}^{K} z_j^k \sum_{\ell=0}^{\ell_j - 1} a_{\ell,j} k^{\ell}, \quad |z_j| = 1,\ a_{\ell,j} \in \mathbb{C},\ \sum_{j=1}^{K} \ell_j = C. \tag{6.1}$$

Denote the overall number of unknown by $R \stackrel{\text{def}}{=} C + K$. Assume that we are given the measurement sequence $\{m_k\}_{k=0,1,\ldots,M}$. Choose an index set $S \subset \{0, 1, \ldots, M\}$ of size exactly R. This defines the so-called "Prony map" $\mathcal{P} : \mathbb{C}^R \to \mathbb{C}^R$, which maps the parameters $\{z_j, a_{\ell,j}\}$ to the measurements $\{m_k\}_{k \in S}$. This also defines the "reconstruction map" $\mathcal{P}^{-1}$, which can be thought of as representing an "ideal reconstruction algorithm". In a small neighborhood of a regular (i.e. noncritical) point of $\mathcal{P}$, the map $\mathcal{P}^{-1}$ is well-defined and well-approximated by its linear part, given by the Jacobian matrix $\mathcal{J}$. Consequently, if the left-hand side of (6.1) is perturbed by a small amount $\varepsilon \ll 1$, then the corresponding perturbation in the values of $\{z_j, a_{\ell,j}\}$ can be easily bounded by the sum of the magnitudes of the entries of the corresponing row of $\mathcal{J}$ times ε.

[3] These systems are important in many problems of mathematics and engineering [2]. They have been used as far back as by Baron de Prony in 1795 [33] for the problem of exponential fitting.

Let the set S be of the form of an arithmetic progression with initial value t and step size σ, i.e.,

$$S = \{t, t+\sigma, t+2\sigma, \dots, t+(R-1)\sigma\}. \tag{6.2}$$

Under the above assumptions, in [6, 10] we have shown that the error for recovering the jump z_j satisfies

$$|\Delta z_j| \leqslant \frac{2}{\ell_j!}\left(\frac{2}{\delta_\sigma}\right)^R \frac{1}{\left|a_{\ell_j-1,j}\right| \sigma^{\ell_j}}\varepsilon, \tag{6.3}$$

where $\delta_\sigma \stackrel{\text{def}}{=} \min_{i\neq j}\left|z_i^\sigma - z_j^\sigma\right|$. A similar, slightly more involved expression is provided for $|\Delta a_{\ell,j}|$.

Now consider the system (2.3). Multiplying both sides by $2\pi(\imath k)^{d+1}$, we obtain the system (6.1) with $\ell_j = d+1$ and perturbation of size $\varepsilon = O\left(M^{-1}\right)$.

Take $S_1 = \{M-(d+2)K+1, \dots, M\}$, which corresponds to the original Eckhoff method of [18]. By (6.3) we get $|\Delta z_j| = O\left(M^{-1}\right)$, i.e., only first-order accuracy. In contrast, for

$$S_2 = \left\{\left\lfloor\frac{M}{(d+2)K}\right\rfloor, 2\left\lfloor\frac{M}{(d+2)K}\right\rfloor, \dots, (d+2)K\left\lfloor\frac{M}{(d+2)K}\right\rfloor\right\}$$

we get $|\Delta z_j| = O\left(M^{-(d+2)}\right)$, i.e., maximal possible asymptotic accuracy. Thus, the Prony systems approach provides another justification for the decimation technique.

But it can provide much more. Indeed, the magnitude of the norm of the Jacobian (bounded from above by (6.3)) provides by definition *the best possible stability bounds* (at least in the case of small perturbations), and therefore the performance (including robustness to noise) of all algorithms (strictly speaking, of those which utilize sampling sets of the form (6.2)) should be compared to these bounds.

To demonstrate this point, consider the decimated Eckhoff algorithm for one point, i.e., Algorithm 2, for the system (3.3), and its stability as provided by Theorem 12. Application of the bound (6.3) to this case gives (here δ_σ is effectively equal to 1, and also $R = (d+2)$, $|a_d| > B^*$, $|\varepsilon| \leqslant R^* \cdot M^{-1}$ and $\sigma = N = \left\lfloor\frac{M}{(d+2)}\right\rfloor$)

$$|\Delta z_j| \leqslant \frac{2^{d+2}}{(d+1)!}\cdot\frac{1}{|a_d| N^{d+1}}\varepsilon < \frac{2^{d+2}(d+2)}{(d+1)!}\cdot\frac{R^*}{B^*}\cdot N^{-d-2}. \tag{6.4}$$

On the other hand, according to the proof of Theorem 12, we have

$$|\Delta\omega| \leqslant C_9(d)\frac{R^*}{B^*}N^{-d-2}.$$

Thus, it can be said that Algorithm 2 provides qualitatively best performance, as both estimates are proportional to $\frac{R^*}{B^*}$. The following calculation provides a simple estimate of the constant C_9.

Proposition 21. *If in step 2 of Algorithm 2 the closest root to the unit circle is chosen, then the constant C_9 satisfies*

$$C_9 \leqslant \frac{3^{d+1}}{(d+1)!}. \tag{6.5}$$

Proof. Using the fact that $w=1$ is a multiple root of s_i^d for $i<d$ and the decomposition (4.3), we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}\,u} p_N^d(u) \bigg|_{u=z} = (d+1)!\alpha_d N^d,$$

and therefore in (4.7) we can take $C_{10} = (d+1)!$. Thus, $C_{12} = 2C_{10} = 2(d+1)!$. To estimate C_{13}, we further have for $|t_\phi| < 2$,

$$\left| e_N^d (t_\phi) \right| \leqslant R^* N^{-1} \underbrace{\sum_{j=0}^{d+1} \binom{d+1}{j} 2^{d+1-j}}_{=3^{d+1}},$$

and thus $C_{13} = 3^{d+1}$. Finally, $C_9 = \frac{2C_{13}}{C_{12}} = \frac{2 \cdot 3^{d+1}}{2(d+1)!}$, which proves (6.5). ☐

The formula (6.3) turns out to be fairly tight, and thus by comparing (6.4) with (6.5) it can be said that Algorithm 2 is away from best accuracy by a factor of

$$\frac{\left(\frac{3}{2}\right)^{d+1}}{2(d+2)}.$$

Similar calculations can be performed for the perturbations in the magnitudes, but due to more complicated expressions we do not present them here.

In order to obtain absolute error bounds for Problem 20 (and for instance compare them with the constants in Theorem 13), the above approach should be extended to handle neighborhoods of finite size, as well as the overdetermined setting (i.e. the case $|S| > R$). We consider this to be an important question for future investigation.

6.2. **Incorrect choice of the smoothness parameter.** An important feature of our method is that the parameters d, K are assumed to be known a priori. Even in the case of one jump, an overestimation of the order d leads to the overall deterioration of the accuracy[4]. Let us briefly show this.

Assume that the function f_j is only piecewise $\tilde{d}$-smooth, i.e., $f_j \in PC\left(\tilde{d}, 1\right)$, when $\tilde{d} < d$, but Algorithm 2 is applied with order d. The formula (4.3) would now read

$$p_N^d(zw) = z^{d+2} \sum_{i=0}^{\tilde{d}} \alpha_i N^i s_i^d(w).$$

Consequently, in the perturbation analysis of Lemma 18, we would have that in a small ε-neighborhood of $z = \omega^N$, the polynomial p_N^d is approximately of magnitude $O\left(N^{\tilde{d}}\right)\varepsilon$. On the other hand, the term ϵ_k in (3.1) is of the order $O\left(k^{-\tilde{d}-2}\right)$, and

[4] In contrast, underestimation might lead to cancellation effects such as the one described in [9]

subsequently the term δ_k in (3.3) is of order $O\left(N^{d-\tilde{d}-1}\right)$. Therefore, the polynomial e_N^d has coefficients of the order $\left|\widetilde{m_{jN}} - m_{jN}\right| = O\left(N^{d-\tilde{d}-1}\right)$. Consequently, the size of the ε-neighborhood containing the perturbed root of q_N^d is in general not better than $\varepsilon = O\left(N^{d-2\tilde{d}-1}\right)$. To conclude, in this case the jump point would be detected with accuracy $O\left(N^{d-2\tilde{d}-2}\right)$ which is of course worse than $O\left(N^{-\tilde{d}-2}\right)$ (the best possible for piecewise $\tilde{d}$-smooth case).

In the general setting of Prony systems (and in Eckhoff's problem in particular), the problem of estimating the model parameters $K, \{\ell_j\}$ from the Fourier data appears to be challenging, especially in the presence of closely spaced jumps and noise. Recent studies (such as [15]) suggest that in any such setting, a crucial role is played by the a priori minimal node separation assumption. On the other hand, the overall degree $\sum \ell_j$ of the Prony system (6.1) can be estimated via the numerical rank of certain Hankel matrices constructed from the data $\{m_k\}$ (see e.g. [34] and references therein), and this information, combined with the node separation assumption, might be used for the correct "clustering". The basis of divided differences might also play an important role in this problem; see [8, 39].

7. Possible extensions

(1) The Eckhoff method has been extended in the literature to handle expansions in other orthonormal basis, such as Chebyshev series ([17, 18]) and Fourier-Jacobi series ([28]). It should be fairly straightforward to extend Algorithm 3 and the analysis of Section 4 to handle these cases.

(2) Another immediate generalization is to the case of piecewise C^∞ functions. By increasing the order d of the reconstruction, according to Theorem 13 the resulting accuracy will eventually be asymptotically smaller than any algebraic power of M. This comes, however, at the cost of the constants of proportionality growing with d.

(3) This last remark brings us to another possible generalization, namely to reconstruction of piecewise-*analytic* functions. One natural line of attack would be to analyze how the constants appearing in the accuracy estimates depend on the smoothness order d (as in the special case provided by Proposition 21), and then choose d in an appropriate way so as to maximize the resulting accuracy (d would be depending on M in this case). According to the results of [1], one may expect (at most) stable root-exponential convergence and unstable exponential convergence. We plan to develop these ideas in a future work.

(4) As noted by K. Eckhoff in [18], the methods can easily be adjusted to handle discontinuities in higher derivatives (and not in the function itself). We expect that decimation will provide the best asymptotic convergence also in these cases.

(5) Extension of the one-dimensional algebraic methods to higher dimensions seems to be highly nontrivial, but nevertheless possible for some special geometric configurations [7, 19]. We consider it to be an important topic for future investigations.

Appendix A. Maximal accuracy for jumps

Proof of Proposition 4. Consider the following subset of $PC(d+1,K)$,

$$B(A,R)=\left\{f\in PC(d+1,K):\ f=\Phi^{(d)}+\Psi;\ |c_k(\Psi)| < R\cdot k^{-d-2};\ \sum_{\ell,j}|a_{\ell,j}|<A\right\}$$

where the smooth part Ψ is in C^{d+1} and the quantities $\{a_{\ell.j}\}$ denote the associated jump magnitudes of the piecewise polynomial $\Phi^{(d)}$ of degree d, as in (2.1).

Let $g\in B(A,R)$ be an arbitrary fixed piecewise polynomial $g=\Phi^{(d)}$ with jumps $\{\xi_1,\dots,\xi_K\}$ and associated jump magnitudes $\{a_{\ell,j}\}$. We will show that there exists an absolute constant C such that for every index M there exists a function $h_M\in B(A,R)$ whose first M Fourier coefficients coincide with those of g, while the corresponding jump locations differ by CM^{-d-2}. Once we show this, it is clear that no deterministic algorithm will be able to reconstruct the jump locations of all functions in $B(A,R)$ with accuracy essentially better than $O\left(M^{-d-2}\right)$.

Denote $\delta=CM^{-d-2}$ where C is to be determined. Let $\Phi_M^{(d)}$ denote another piecewise polynomial of degree d with jumps

$$\{\eta_1=\xi_1+\delta,\dots,\eta_K=\xi_K+\delta\}$$

and the same jump magnitudes $\{a_{\ell,j}\}$ as those of $\Phi^{(d)}$. Let

$$b_k=c_k\left(\Phi^{(d)}-\Phi_M^{(d)}\right).$$

Finally, take

$$h_M(x)\stackrel{\text{def}}{=}\Phi_M^{(d)}(x)+\sum_{|k|=0}^{M}b_k\,\mathrm{e}^{\imath kx}.$$

Clearly, $c_k(g)=c_k(h_M)$ for $|k|=0,1,\dots,M$. In order to ensure that $h_M\in B(A,R)$ we need to choose C small enough such that

$$|b_k|\le R\cdot k^{-d-2};\qquad k=1,2,\dots,M.$$

Let us show that $C\stackrel{\text{def}}{=}\frac{2\pi R}{A}$ satisfies the above condition. Indeed,

$$\begin{aligned} b_k=c_k\left(\Phi^{(d)}-\Phi_M^{(d)}\right) &= \frac{1}{2\pi}\sum_{j=1}^{K}\mathrm{e}^{-\imath\xi_j k}\sum_{\ell=0}^{d}\frac{a_{\ell,j}}{(\imath k)^{\ell+1}}-\frac{1}{2\pi}\sum_{j=1}^{K}\mathrm{e}^{-\imath\eta_j k}\sum_{\ell=0}^{d}\frac{a_{\ell,j}}{(\imath k)^{\ell+1}}\\ &= \left(\mathrm{e}^{\imath\delta k}-1\right)\frac{1}{2\pi}\sum_{j=1}^{K}\mathrm{e}^{-\imath\eta_j k}\sum_{\ell=0}^{d}\frac{a_{\ell,j}}{(\imath k)^{\ell+1}}. \end{aligned}$$

Now obviously

$$\left|\frac{1}{2\pi}\sum_{j=1}^{K}\mathrm{e}^{-\imath\eta_j k}\sum_{\ell=0}^{d}\frac{a_{\ell,j}}{(\imath k)^{\ell+1}}\right|\le\frac{A}{2\pi k}.$$

From geometric considerations we have $\left|1-\mathrm{e}^{\imath\delta k}\right|\leq\delta k$, therefore,

$$|b_k| \quad\leq\quad \frac{A}{2\pi k}\delta k=\frac{A}{2\pi}CM^{-d-2}=\frac{A}{2\pi}\cdot\frac{2\pi R}{A}\cdot M^{-d-2}<RM^{-d-2}.$$

This completes the proof. □

Acknowledgements

The author would like to thank B. Adcock and Y. Yomdin for useful discussions. The author is also very grateful to the reviewer for many constructive suggestions.

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