ALGEBRAIC RECONSTRUCTION OF PIECEWISE-SMOOTH FUNCTIONS FROM FOURIER DATA

Dmitry Batenkov, Yosef Yomdin

Department of Mathematics,
Weizmann Institute of Science, Israel
Emails: {dima.batenkov, yosef.yomdin}@weizmann.ac.il

ABSTRACT

This paper presents some recent progress on the problem of reconstructing piecewise-smooth functions with unknown singularity locations from Fourier measurements, using the methodology of Algebraic Signal Sampling - both on uniform and non-uniform grids. We describe an explicit reconstruction algorithm which can recover both the locations of the singularities and the pointwise values of the function with accuracy which is "half as accurate" compared to the "classical" approximation theory for smooth functions.

Keywords— Algebraic Sampling, Fourier inversion, non-linear approximation, piecewise-smooth functions, Gibbs phenomenon, non-uniform sampling

1. INTRODUCTION

Consider the problem of reconstructing a piecewise-smooth function $f:[-\pi,\pi]\to\mathbb{R}$ with a finite number of jump discontinuities at unknown locations from a finite number of its Fourier samples

$$c_k(f)\stackrel{\text{def}}{=}\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\,\mathrm{e}^{-\imath kt}\,\mathrm{d}\,t,\qquad k=0,1,\dots,N.$$

This problem is very important in many applications (such as spectral methods in PDEs, tomographic reconstruction and others, see e.g. [12]). For smooth functions, simple summation of the Fourier series provides very good reconstruction. For example, if $f\in C^d$ then the (pointwise) reconstruction error from the first N Fourier coefficients is of the order N^{-d}. However, the presence of jumps leads to slow convergence of the Fourier series, and as a result the reconstruction error is often very large (especially near the jumps, this effect commonly being known as the "Gibbs phenomenon"). Possibility of an accurate reconstruction also in the presence of jumps is being extensively investigated up to this day - see for example [2, 8, 9, 12, 15, 17] (more comprehensive list may be found in [6]). It is generally accepted that the most difficult remaining problem is the accurate determination of the jump locations, or "edges", from the Fourier data. We have previously conjectured ([5, 10]) that *there is a nonlinear algebraic procedure reconstructing any signal in the class of piecewise C^d functions from its first N coefficients with the overall accuracy of order N^{-d}, including the jump positions as well as the smooth pieces.*

Our main result is as follows:

Theorem 1. *Let $f:[-\pi,\pi]\to\mathbb{R}$ have K jump discontinuities $\{\xi_j\}_{j=1}^K$, and let it be d_1-times continuously differentiable between the jumps.*

Then for every integer d satisfying $2d+1\le d_1$, there exist explicit constants C_1,K_1 depending on several a-priori bounds (concerning the geometry and the magnitudes of the jumps) such that for all $N>K_1$ one can reconstruct the locations of the jumps with error $\sim N^{-d-2}$, and the whole f with the pointwise accuracy

$$\left|\widetilde{f}(y)-f(y)\right|\le C_1\cdot N^{-d-1}$$

from the first N Fourier coefficients of f.

In other words, *the "jumps" (and subsequently the pointwise values) can be reconstructed with at least **half** the conjectured accuracy*. We present an explicit reconstruction algorithm and provide estimates for its accuracy (full details will be published elsewhere, see [6]). We also present numerical evidence which both confirms the theoretical estimates and also suggests that the attained asymptotic accuracy might be the best possible one.

We discuss also the reconstruction problem for non-uniform Fourier samples in Section 5.

Our approach can be regarded as a kind of "Algebraic Signal Sampling", which is applicable to situations where the signal has certain "simple" structure, i.e. can be represented by a small number of parameters ([5, 16]). Some other examples of this general approach include: signals with finite rate of innovation ([7, 18]), reconstruction of planar shapes from real and complex moments ([4, 13]), and also piecewise D-finite moment inversion and reconstructing combinations of shifts of a given function ([3, 5]).

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2. ALGEBRAIC FOURIER RECONSTRUCTION

Assume the notation of Theorem 1. Denote the associated jump magnitudes of f at ξ_j by

$$A_{l,j} \stackrel{\text{def}}{=} f^{(l)}(\xi_j^+) - f^{(l)}(\xi_j^-)$$

We write the piecewise smooth f as the sum $f = \Psi + \Phi$, where $\Psi(x)$ is smooth and periodic and $\Phi(x)$ is a piecewise polynomial of degree d, uniquely determined by $\{\xi_j\}, \{A_{i,j}\}$ such that it "absorbs" all the discontinuities of f and its first d derivatives. If we knew what Φ is, we could then reconstruct Ψ from the known quantities $\{c_k(f) - c_k(\Phi)\}$ very accurately. This idea (known as "convergence acceleration") is very old and goes back at least to A.N.Krylov ([14]). But what if Φ is unknown a-priori? A key observation (first proposed by K.Eckhoff in [8]) is that if Ψ is sufficiently smooth, then the contribution of $c_k(\Psi)$ to $c_k(f)$ is negligible **for large** k. Therefore, for some large enough $M < N$ one can write an approximate equality

$$c_k(f) \approx c_k(\Phi) \quad k \geq M.$$

Substituting $\{\xi_j\}, \{A_{i,j}\}$ into the expression for $c_k(\Phi)$ yields the following system of algebraic equations (see [6, 8]):

$$c_k(f) \approx \frac{1}{2\pi} \sum_{j=1}^{K} \omega_j^k \sum_{l=0}^{d} \frac{A_{l,j}}{(\imath k)^{l+1}} \qquad k = M, M+1, \ldots, N \tag{1}$$

where $\omega_j \stackrel{\text{def}}{=} \mathrm{e}^{-\imath \xi_j}$. Solving this system with high accuracy will give us an accurate approximation for Φ, and, subsequently, for f itself.

Variations of the system (1) appear also in [2, 8, 15] in similar contexts. Our solution method, outlined below, is different from the methods presented there, and it allows for explicit accuracy analysis in the most generic case. Our approach can be summarized as follows:

1. Obtain approximate positions of the jumps (using for example Eckhoff's method of order zero - [8]).
2. Localize each ξ_j by convolving the original function in the Fourier domain with a C^∞ "bump" function centered at the above approximate jump position.
3. Solve the system (1) separately for each ξ_j *with high accuracy.*

The first two steps are fairly straightforward, thus we shall not elaborate them here - full details are to be found in [6]. Here let us only mention that these steps do not destroy the asymptotic estimates of item 3 above. The last step is described in Section 3 below. The key idea is to decouple the reconstruction order (the integer d in (1)) from the real smoothness of f - the actual number of continuous derivatives (the integer d_1 in Theorem 1). Once we assume that $d_1 > d$, the error term in the left-hand side of (1) has an additional structure (see (6) below), resulting in remarkable cancellations in the solution.

3. SINGLE JUMP

Let d be fixed. As explained above, we can now consider the system (1) for a single jump point:

$$c_k(f) \approx \frac{\omega^k}{2\pi} \sum_{l=0}^{d} \frac{A_l}{(\imath k)^{l+1}} \qquad k = M, \ldots, M+d+1. \tag{2}$$

For simplicity, we take the number of equations to be equal to the number of unknowns. (It is not clear that this choice is optimal.) Now we consider the system (2) to be a perturbation of the exact system

$$c_k(\Phi) = \frac{\omega^k}{2\pi} \sum_{l=0}^{d} \frac{A_l}{(\imath k)^{l+1}} \qquad k = M, \ldots, M+d+1. \tag{3}$$

in the sense that solutions to (3) are perturbations of solutions of (2). Eliminating $\{A_0, \ldots, A_d\}$ from (3) gives us for each $k \in \mathbb{N}$ a single equation $p_k(\omega) = 0$ where $p_k(z)$ is a polynomial of degree $d+1$ defined as follows:

$$\begin{aligned} m_k &\stackrel{\text{def}}{=} 2\pi(\imath k)^{d+1} c_k(\Phi) = \omega^k \sum_{l=0}^{d} (\imath k)^{d-l} A_l \\ p_k(z) &\stackrel{\text{def}}{=} \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} m_{k+j} z^{d+1-j} \end{aligned} \tag{4}$$

The values m_k are unknown, but we can approximate them with the known quantities $r_k \stackrel{\text{def}}{=} 2\pi\, (\imath k)^{d+1}\, c_k(f)$. Then we define

$$q_k(z) \stackrel{\text{def}}{=} \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} r_{k+j} z^{d+1-j} \tag{5}$$

As $k \to \infty$, we know that $r_k \to m_k$ and therefore the roots of $q_k(z)$ approach those of $p_k(z)$. One of these latter roots is ω, which is precisely the value we seek. Those two sequences of polynomials subsequently become our main objects of study.

Assume that f has in fact $d_1 \geq d$ continuous derivatives everywhere in $[-\pi, \pi] \setminus \{\xi\}$. Then the Fourier coefficients of f can be written as

$$c_k(f) = c_k(\Phi) + \frac{\omega^k}{2\pi} \sum_{l=d+1}^{d_1} \frac{A_l}{(\imath k)^{l+1}} + c_k(\Psi^*) \tag{6}$$

such that $|c_k(\Psi^*)| \leq R^* k^{-d_1-2}$ for some constant R^*. Denote

$$H \stackrel{\text{def}}{=} \max\left(2, \sum_{l=0}^{d_1} |A_l|\right) + R^*$$

Then we have the following result.

Theorem 2. *Let f have $d_1 \geq 2d+1$ continuous derivatives everywhere in $[-\pi, \pi] \setminus \{\xi\}$. Let $q_k(z)$ be as defined in (5), and let $\left\{\kappa_i^{(k)}\right\}_{i=0}^{d}$ denote its roots, such that $\left|\kappa_0^{(k)}\right| \leq \ldots \left|\kappa_d^{(k)}\right|$. Let $\{\phi_i\}_{i=1}^{d}$ denote the roots of the Laguerre polynomial $\mathcal{L}_d^{(1)}$,* *such that* $|\phi_1| < \dots |\phi_d|$. *Let* $y_0^{(k)} = \omega$ *and* $y_i^{(k)} = \frac{\omega}{1-\frac{\phi_i}{k}}$ *for* $i = 1, \dots, d$. *Then there exist constants* C_2, C_3, C_4 *and* K_2 *such that for every* $k > K_2 H$ *the following statemenets are true:*

1. *The numbers* $\left\{y_i^{(k)}\right\}$ *lie on the ray* $O\omega$, *so that* $\left|y_i^{(k)}\right| \geq 1$, *and they are sufficiently separated:*

$$C_2 k^{-1} \leq \left|y_i^{(k)} - y_j^{(k)}\right| \qquad 0 \leq i < j \leq d$$

2. *Each of the numbers* $\left\{\kappa_i^{(k)}\right\}_{i=1}^{d}$ *is close to some* $y_i^{(k)}$:

$$\left|\kappa_i^{(k)} - y_i^{(k)}\right| \leq C_3 \cdot H \cdot k^{-2}$$

3. *The smallest* $\kappa_0^{(k)}$ *is very close to* ω:

$$\left|\kappa_0^{(k)} - \omega\right| \leq C_4 \cdot H \cdot k^{-d-2}$$

Proof outline. We apply perturbation analysis to $q_k(z)$, using Rouche's theorem from complex analysis. First we write $q_k(z) = p_k(z) + e_k(z)$. Then we consider small disks around each one of the roots of p_k, whose diameters shrink as $k \to \infty$, and such that $|p_k(z)| > |e_k(z)|$ for all z on the boundaries of these disks. Then we conclude that q_k has a root which is close to the corresponding root of p_k. The roots of p_k, in turn, lie close to the numbers $y_i^{(k)}$ by similar reasoning - the polynomials p_k turn out to be certain perturbations of generalized Laguerre polynomials ([1, Chapter 22]). □

The immediate consequence of Theorem 2 is that for every $M > K_2 H$, the root of $q_M(z)$ which is closest to the unit circle will provide an approximation to ω up to order M^{-d-2}. Now this approximate value can be substituted back into (2), which becomes a linear system with respect to the unknowns $A_0, \dots, A_d$. The error in the left-hand side of this system is again given by (6). Subsequently, the error in the solution is given by the following theorem.

Theorem 3. *Assume that* $d_1 \geq 2d + 1$ *and* $k > K_2 H$, *so that by Theorem 2 we have* $\left|\widetilde{\omega}_{(k)} - \omega\right| \leq C_4 \cdot H \cdot k^{-d-2}$. *Then there exist constants* C_5, K_3 *such that for every* $k > K_3 H$ *and* $l = 0, 1, \dots d$ *the error in determining* A_l *is*

$$\left|\widetilde{A}_l^{(k)} - A_l\right| \leq C_5 \cdot H^2 \cdot k^{l-d-1}$$

4. NUMERICAL EXPERIMENTS

We have performed a number of experiments for reconstructing a function with a single jump. The jumps locations, magnitudes and the smooth pieces were chosen randomly. The results are presented in Figures 1 and 2. The optimality of $d = \frac{d_1}{2} - 1$, as well as the asymptotic order of convergence, are clearly seen to fit the theoretical predictions. The instability and eventual breakup of the measured accuracy for large values of M is due to the finite-precision calculations.

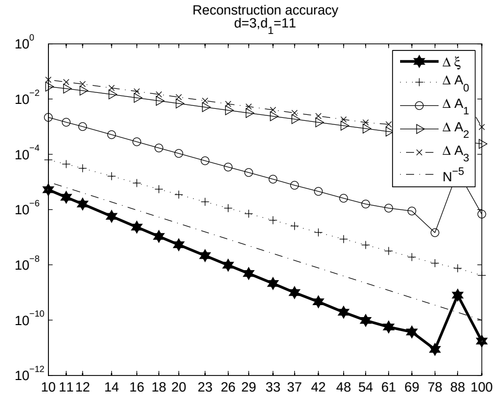

Fig. 1. Reconstruction of a single jump. Accuracy of reconstruction with $d = 3$ and $d_1 = 11$, as a function of M.

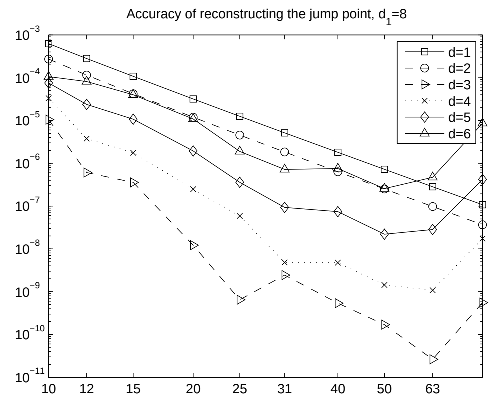

Fig. 2. Dependence of the accuracy on the order with fixed smoothness $d_1 = 8$, with increasing M.

The case of more than one jump was also investigated numerically and the results fit the theoretical predictions as well, but this requires a separate discussion.

5. NON-UNIFORM MEASUREMENTS

Fourier coefficients $c_k(f)$ of f can be considered as samples of the Fourier transform $\mathcal{F}(f)$ at the integer points. An important problem is to extend the results above to the non-uniform samples of $\mathcal{F}(f)$. So we fix a "sampling set" $Z = \{s_1, s_2, \dots\}$, $s_j \in \mathbb{R}$ and get as the "measurements" the generalized Fourier coefficients

$$c_{s_j}(f) = \int_{-\infty}^{\infty} \mathrm{e}^{-2\pi\imath s_j x} f(x)\,\mathrm{d}\,x.$$

To simplify the presentation consider "signals" of the form $f(x) = \sum_{i=1}^{n} A_i \delta(x - x_i)$. Reconstruction of piecewise smooth functions from non-uniform Fourier measurements can be treated in a similar way and we plan to present it separately.

We utilize the fact that for f a linear combination of δ-functions its Fourier transform $\mathcal{F}(f)(s)$ is an exponential polynomial $\sum_{i=1}^{n} A_i \, \mathrm{e}^{-2\pi\imath s x_i}$. Then we use a "discrete" version of the classical Turan-Nazarov inequality recently obtained in [11, 19], and, in particular, a metric invariant $\omega(Z)$ introduced there using metric entropy of Z. We obtain the following result:

Theorem 4. *Assume that* $\omega(Z) > 0$. *Then for any* f *as above the usual Fourier coefficients* $c_k(f),\ |k| \le N$ *can be uniquely reconstructed from the samples* $c_{s_j}(f)$ *for* $s_j \in Z$. *The uniform norm of the reconstruction operator is bounded by* $\mathrm{e}^{2N}\left(\frac{4N}{\omega(Z)}\right)^n$.

The invariant $\omega(Z)$ usually can be accurately estimated in explicit geometric terms. In particular, this can be done for Z arising as zeroes of the Fourier transform of the shifted signals, as it appears in the decoupling procedure described in [5] for linear combinations of shifts of several signals.

6. CONCLUSIONS AND FUTURE WORK

We have demonstrated that the algebraic reconstruction methods can achieve at least "half the classical" accuracy in the Fourier inversion problem for piecewise-smooth data. An important question remains: is this accuracy the best possible one?

The algebraic system (1) appears (in various disguises) in several nonlinear reconstruction methods in Signal Processing ([3, 4, 5, 7, 13, 18]). We hope that our results might be relevant to those methods as well.

7. REFERENCES

[1] M. Abramowitz and I.A. Stegun. Handbook of mathematical functions: with formulas, graphs, and mathematical tables. 1965.

[2] N.S. Banerjee and J.F. Geer. Exponentially accurate approximations to periodic Lipschitz functions based on Fourier series partial sums. *Journal of Scientific Computing*, 13(4):419–460, 1998.

[3] D. Batenkov. Moment inversion problem for piecewise D-finite functions. *Inverse Problems*, 25(10):105001, October 2009.

[4] D. Batenkov, V. Golubyatnikov, and Y. Yomdin. On one nonlinear problem of reconstructing a planar region with singular boundaries from a finite number of measurements (in Russian). *Department of Mathematical Analysis, Gorno-Altayskiy Univ., Russia*, 2:17 – 23, 2010.

[5] D. Batenkov, N. Sarig, and Y. Yomdin. An "algebraic" reconstruction of piecewise-smooth functions from integral measurements. *Proc. of Sampling Theory and Applications (SAMPTA)*, 2009.

[6] D. Batenkov and Y. Yomdin. Algebraic Fourier reconstruction of piecewise smooth functions, submitted. Arxiv preprint arXiv:1005.1884.

[7] P.L. Dragotti, M. Vetterli, and T. Blu. Sampling Moments and Reconstructing Signals of Finite Rate of Innovation: Shannon meets Strang-Fix. *IEEE Transactions on Signal Processing*, 55(5):1741, 2007.

[8] K.S. Eckhoff. Accurate reconstructions of functions of finite regularity from truncated Fourier series expansions. *Mathematics of Computation*, 64(210):671–690, 1995.

[9] K.S. Eckhoff. On a high order numerical method for functions with singularities. *Mathematics of Computation*, 67(223):1063–1088, 1998.

[10] B. Ettinger, N. Sarig, and Y. Yomdin. Linear versus Non-Linear Acquisition of Step-Functions. *Journal of Geometric Analysis*, 18(2):369–399, 2008.

[11] O. Friedland, Y. Yomdin, Turan-Nazarov Inequality for Discrete Sets, in preparation.

[12] D. Gottlieb and C.W. Shu. On the Gibbs phenomenon and its resolution. *SIAM Review*, pages 644–668, 1997.

[13] B. Gustafsson, C. He, P. Milanfar, and M. Putinar. Reconstructing planar domains from their moments. *Inverse Problems*, 16(4):1053–1070, 2000.

[14] L.V. Kantorovich and V.I. Krylov. *Approximate Methods of Higher Analysis*. Fizmatgiz, Moscow, 1962.

[15] G. Kvernadze. Approximation of the discontinuities of a function by its classical orthogonal polynomial Fourier coefficients. *Math. Comp*, 79:2265–2285, 2010.

[16] N. Sarig and Y. Yomdin. Signal Acquisition from Measurements via Non-Linear Models. *Mathematical Reports of the Academy of Science of the Royal Society of Canada*, 29(4):97–114, 2008.

[17] E. Tadmor. Filters, mollifiers and the computation of the Gibbs phenomenon. *Acta Numerica*, 16:305–378, 2007.

[18] M. Vetterli, P. Marziliano, and T. Blu. Sampling signals with finite rate of innovation. *IEEE Transactions on Signal Processing*, 50(6):1417–1428, 2002.

[19] Y. Yomdin, Remez Inequality for Discrete Sets, to appear, Israel J. of Math.