Prony Systems via Decimation and Homotopy Continuation

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ABSTRACT

We consider polynomial systems of Prony type, appearing in many areas of mathematics. Their robust numerical solution is considered to be difficult, especially in "near-colliding" situations. We transform the nonlinear part of the Prony system into a Hankel-type polynomial system. Combining this representation with a recently discovered "decimation" technique, we present an algorithm which applies homotopy continuation on a sequence of modified Hankel-type systems as above. In this way, we are able to solve for the nonlinear variables of the original system with high accuracy when the data is perturbed.

Categories and Subject Descriptors

G.1.0 [**General**]: Conditioning (and ill-conditioning); G.1.5 [**Roots of Nonlinear Equations**]: Continuation (homotopy) methods

General Terms

Algorithms

Keywords

Prony systems, decimation

1. INTRODUCTION

Consider the following approximate algebraic problem.

Problem 1. *Given $(\tilde{m}_0, \dots, \tilde{m}_{N-1}) \in \mathbb{C}^N$, find $s \in \mathbb{N}$, a multiplicity vector $D = (d_1, \dots, d_s) \in \mathbb{N}^s$ with $d := \sum_{j=1}^s d_j$ and $2d < N$, and complex numbers $\{z_j, \{a_{\ell,j}\}_{\ell=0}^{d_j-1}\}_{j=1}^s$ with $a_{d_j-1,j} \neq 0$ such that for some perturbation vector $(\epsilon_k) \in \mathbb{C}^N$ with $|\epsilon_k| < \varepsilon$ we have*

$$\tilde{m}_k = \underbrace{\sum_{j=1}^s z_j^k \sum_{\ell=0}^{d_j-1} a_{\ell,j} k^\ell}_{:=m_k} + \epsilon_k, \qquad k = 0, \dots, N-1. \tag{1}$$

SNC'14, July 28-31, 2014, Shanghai, China.

http://dx.doi.org/10.1145/2631948.2631961

This so-called "Prony problem" appears in signal processing, frequency estimation, exponential fitting, Padé approximation, sparse polynomial interpolation, spectral edge detection, inverse moment problems and recently in theory of super-resolution (see [2, 4, 7, 8] and references therein).

The high degree of symmetry in the system of equations (1) allows to separate the problem into a linear and a nonlinear part. The basic observation (due to Baron de Prony [12]) is that the sequence of exact measurements $\{m_k\}$ satisfies a linear recurrence relation

$$\sum_{\ell=0}^d m_{k+\ell} c_\ell = 0, \qquad k \in \mathbb{N}, \tag{2}$$

where $\{c_\ell\}$ are defined by $\prod_{j=1}^s (z - z_j)^{d_j} \equiv \sum_{\ell=0}^d c_\ell z^\ell$. Based on the above observation, several algorithms have been proposed for recovering the "nodes" $\{z_j\}_{j=1}^s$, such as MUSIC/ESPRIT, matrix pencils and Variable Projections (VARPRO). While the majority of these algorithms perform well on simple and well-separated nodes (i.e. with $D = (1, 1, \dots, 1)$), they are poorly adapted to handle either multiple/clustered nodes, non-Gaussian noise or large values of N ([6, 10]).

Robust detection of these near-singular situations, i.e. correct identification of the collision pattern D is, therefore, one of the most important questions of interest. While the integer d can be estimated via numerical rank computation of the so-called "data matrix" (see [4, 5]), the determination of D is a more delicate task, which requires an accurate estimation of the distance from the data point to the nearest "pejorative" manifold of larger multiplicity, and comparing it with the a-priori bound ε on the error. We hope that the present (and future) symbolic-numeric techniques such as [11] will eventually provide a satisfactory answer to this question.

Once D is determined, it remains to actually solve the system with maximal possible accuracy. In what follows we concentrate on this latter task and leave the more challenging problem of multiplicity pattern detection to future research.

We assume throughout that $|z_j| = 1$.

2. REGULARIZATION VIA DECIMATION

In many applications, the number of available measurements N can be larger than the "problem size" $R := d + s$. Thus an important question arises: *how to efficiently utilize the additional measurements?* On one hand, methods such as ESPRIT compute an SVD on the full data matrix (of size $N \times R$), which requires $O\left(N^2 R\right)$ operations, and this can become prohibitive for large N. On the other hand, since the recurrence relation (2) is valid for all $k \in \mathbb{N}$, another possibility is to solve a sequence of square problems (by whatever method) with consecutive data segments $\{m_k, \dots, m_{k+R-1}\}_{k=0}^{N-R}$. However, in [3] we show that it is much more preferable from the numerical point of view to consider "decimated" sequences $\{m_0, m_p, m_{2p}, \dots, m_{(R-1)p}\}$ for appropriate choices of the "decimation parameter" $p \in \mathbb{N}$. In particular, the condition number of the node z_j (estimated via the corresponding data-result mapping [13]) is shown to be proportional to p^{-d_j}. When $N \gg 1$, the decimation parameter can be taken to be arbitrarily large as well. Based on rigorous analysis of the case $s = 1$ in the special setting of [2], and initial numerical verifications in the general case, we conjecture that the decimation technique provides asymptotically (w.r.t. N) accurate approximation to the full overdetermined problem. Apparently, when $N \to \infty$, decimation with parameter p turns a poorly conditioned multiple root z_j into a collection of well-separated, well-conditioned simple roots, one of which is z_j^p.

3. RECONSTRUCTION ALGORITHM

We present a novel algorithm for recovering the nodes $\{z_j\}$ from the perturbed measurements $\{\tilde{m}_k\}_{k=0}^{N-1}$, consisting of two main ingredients.

First, starting from (2) and writing the c_ℓ's via Vieta's formulas, we arrive at the following relation for all $k \in \mathbb{N}$:

$$\sum_{i=0}^{d} m_{k+i}\sigma_{d-i}(\underbrace{z_1, \dots, z_1}_{\times d_1}, \dots, \underbrace{z_s, \dots, z_s}_{\times d_s}) = 0 \qquad (3)$$

with respect to the unknowns $z_1, \dots, z_s$, where $\sigma_j(\dots)$ is the elementary symmetric function of order j in d variables[1].

Then, we apply decimation technique to (3) and construct a sequence $\mathcal{H}_p$ of decimated $s \times s$ square Hankel-type systems from the perturbed measurements $\{\tilde{m}_k\}$:

$$\mathcal{H}_p : \left\{ \sum_{i=0}^{d} \tilde{m}_{k+ip}\sigma_{d-i}\left(z_1^p, \dots, z_1^p, \dots, z_s^p, \dots, z_s^p\right) = 0 \right\}_{k=0,p,\dots,p(s-1)} . \qquad (4)$$

The algorithm proceeds as follows:

1. Input: D, approximations to $\{z_j\}$, cutoff parameter ρ, the sequence $\{\tilde{m}_k\}$.
2. Choose appropriate decimation parameters p which locally minimize the condition number (computed using the initial approximations).
3. For each p as above, solve $\mathcal{H}_p$ via homotopy continuation for the variables $w_1 = z_1^p, \dots, w_s = z_s^p$.
4. For each computed solution $\tilde{\mathbf{z}}_\mathbf{p} = (\tilde{z}_1^p, \dots, \tilde{z}_s^p)$, choose among the possible p approximations for each component the ones which fall within distance ρ from the initial approximation.

4. NUMERICAL RESULTS

The proposed Decimated Homotopy (DH) algorithm has been implemented using MATLAB's interface to PHCPACK Release 2.3.87 [9]. We have compared its performance with generalized ESPRIT algorithm [1]. The results can be summarized as follows (see an example in Figure 1):

[1]This transformation is essentially equivalent to elimination of the linear variables $a_{\ell,j}$ from (1), written in the basis of finite differences [4].

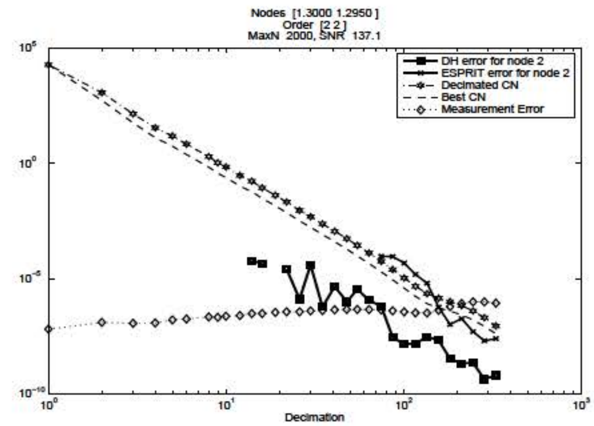

Figure 1: A sample run of DH vs ESPRIT. (CN is Condition Number.)

1. The accuracy of DH is comparable with, and sometimes surpasses ESPRIT by 1-2 significant digits.
2. DH achieves desired accuracy in larger number of cases.
3. The running time of each iteration of DH is constant (depending only on the number of parameters R), while running time of ESPRIT grows with N.

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