Topics in Inverse Problems

Course summary & projects
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Recap

Part I: Linear Inverse Problems

- Well-posed problems: Hadamard's criteria, examples
- Integral equations (1st kind) in a Hilbert space setting
- Picard's theorem: conditions for solvability of $Ax = y$ when $A : X \to Y$ is a compact operator
- Regularization: using a-priori information for restoring **stability**
  - (best) a-priori stability estimates
  - "generic" regularisation scheme: "frequency" filters, convergence
  - Tikhonov regularisation
  - A-posteriori parameter selection: L-curve, discrepancy principle
- Discretization: quadrature, collocation, Galerkin
- Iterative methods: Landweber iteration, conjugate gradient

Part II: (Classical) Resolution & Super-resolution

- Nyquist sampling theorem
- Measures of resolution
    - “Highest frequency”: decay of singular values of the forward operator
    - mildly vs severely ill-posed problems
    - Zero-spacings (oscillations) of singular functions
- Bandlimited systems and convolution operators
    - Frequency domain regularisation
    - Distribution of eigenvalues, Szego’s theorems
    - Ideal low-pass kernel (sinc / Slepian operator): singular system
    - SR for space-limited objects: logarithmic improvement in resolution
- Super-resolution and analytic continuation
    - A-priori stability via potential theory
    - General approach via expansions, choosing optimal truncation parameter
    - Paley-Wiener theorem
    - Speed of convergence of the trapezoidal rule

Part III: parametric super-resolution

- Non-smooth problems: linear methods not accurate enough
- Detection of singularities in piecewise-smooth functions: an algebraic method. Optimal dependence on smoothness between jumps.
- Super-resolution of point sources: $f(t) \sim \sum c_j \delta(t - t_j)$ from moments $m_k = \hat{f}(k),\ k = 0,1,\ldots,N$ and additive bounded noise $\|e\| \leq \epsilon$
  - Existence&uniqueness: Hankel matrices, Pade approximation
  - Local stability: Jacobians, Vandermonde matrices, Hermite interpolation.
  - Decimation: selecting a well-behaved subset of equations
  - Semi-local stability: bound on ϵ. Quantitative inverse function theorem.
  - Dependence of stability on the minimal separation Δ, bandwidth N and clustering
  - Phase transition $N \approx 1/\Delta$
  - Minimax framework: best worst-case error and oracle algorithm
  - Stability of least-squares

D. Batenkov, G. Goldman, and Y. Yomdin, "Super-resolution of near-colliding point sources," *Arxiv:1904.09186 [math]*, 2019.

D. Batenkov, "Stability and super-resolution of generalized spike recovery," *Applied and computational harmonic analysis*, vol. 45, iss. 2, pp. 299-323, 2018.

D. Batenkov, "Complete Algebraic Reconstruction of Piecewise-Smooth Functions from Fourier Data," *Mathematics of Computation*, vol. 84, iss. 295, p. 2329–2350, 2015.

D. Batenkov and Y. Yomdin, "On the accuracy of solving confluent Prony systems," *SIAM J. Appl. Math.*, vol. 73, iss. 1, p. 134–154, 2013.

Projects

General guidelines

- Groups of 2-3
- Singles are OK but require prior confirmation
- Requirements: final report + code + presentation
- Structure of the report
    - Introduction: general context, motivation
    - Summary of relevant paper (s)
    - Optional chapters (e.g. **your** numerical experiments)
    - Formal references
- Report in English (LaTeX), 10-15 pages, code in any programming language
- Document everything! You are evaluated mainly for the effort (but I have to see some proof)
- Timeline:
    - July 15th: send me an email incl. group members and chosen topic (but note that it is first-come first serve)
    - November 1st: submit final report + code
    - Presentation: schedule a meeting within 1 month of the submission deadline
- **90% of the final grade**

Possibility 1: a particular inverse problem of your choice

- Describe the problem (some physics etc.)
- Formulate as an operator
- Demonstrate ill-posedness/ill-conditioning
- Implement a regularisation scheme with parameter selection
- Use simulated (beware of inverse crime) or real data (if you can get it)
- Study resolution

Possible sources:

- Bertero book Chapter 8: X-ray tomography, emission tomography, inverse diffraction, inverse source, linearised inverse scattering
- Hansen's IRTools: deblurring, inverse diffusion, NMR relaxometry, tomography
- Handbook of math. Methods in imaging book - some chapter from part II
- Geophysics: chapters from parts III-V
- … (a problem you care about)

Other possibilities

1. Open problems mentioned in class
   - May be a jump-start for M.Sc. thesis
   - Talk to me to decide scope
2. A paper from the list
   - Understand, explain, implement algorithm (should be not too difficult in most cases)
   - Study the (super-) resolution properties of the algorithm, comparing to optimal bounds when possible
3. Suggest a topic