Padé approximation

Given $f(z) = \sum_{k=0}^{\infty} m_k z^{-k-1}$, construct

(**) $\quad R(z) = \frac{P(z)}{Q(z)} \qquad \deg P < \deg Q \leq d$

s.t. $\quad \frac{P(z)}{Q(z)} - f(z) = O\left(z^{-(2d+1)}\right) \qquad (1)$

Already seen that (*) is solvable if (**) is solvable.

Let's multiply by $Q(z)$

(2) $\quad Q(z) f(z) - P(z) = O\left(z^{-(d+1)}\right)$

$$= \frac{e_1}{z^{d+1}} + O\left(z^{-(d+2)}\right) \dots$$

$P(z) = b_0 + \dots + b_{d-1} z^{d-1}$

$Q(z) = c_0 + \dots + c_d z^d$

$$\left(c_0 + c_1 z + \dots + c_d z^d\right)\left(\frac{m_0}{z} + \frac{m_1}{z^2} + \dots\right) - \left(b_0 + b_1 z + \dots + b_{d-1} z^{d-1}\right)$$

$$= \frac{e_1}{z^{d+1}} + \dots$$

$$\mathrm{II}\begin{cases} z^{d-1}: & c_d m_0 - b_{d-1} = 0 \\ z^{d-2}: & c_d m_1 + c_{d-1} m_0 - b_{d-2} = 0 \\ \vdots & \\ z^0: & c_1 m_0 + c_2 m_1 + \ldots + c_d m_{d-1} - b_0 = 0 \end{cases}$$

$$\begin{bmatrix} & & & m_0 \\ & & m_0 & m_1 \\ & \iddots & & \vdots \\ m_0 & \ldots & & m_{d-1} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ \vdots \\ c_d \end{bmatrix} = \begin{bmatrix} b_{d-1} \\ \vdots \\ \vdots \\ b_0 \end{bmatrix}$$

$$\mathrm{I}\begin{cases} z^{-1}: & c_0 m_0 + c_1 m_1 + \ldots + c_d m_d = 0 \\ \vdots & \\ z^{-d}: & c_0 m_{d-1} + \ldots + c_d m_{2d-1} = 0 \end{cases}$$

$$\begin{bmatrix} m_0 & \ldots & m_d \\ m_1 & & \\ \vdots & & \\ m_{d-1} & \ldots & m_{2d-1} \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_d \end{bmatrix} = 0$$

If $c_{r+1} = c_{r+2} = \ldots = c_d = 0$ $\quad (\deg Q \le r)$ $\qquad \tilde{H}_d$

$\Rightarrow \deg P \le r-1$

$\Rightarrow$ (2) always has a solution.

Important observation: (2) does not always imply (1),

only if $q_d \neq 0$

$\Rightarrow$ (1) might be unsolvable.

Suppose it is solvable, then we can express P/Q as a sum of elementary fractions.

If $Q(z) = \prod_{j=1}^{s} (z - t_j)^{d_j}$ $\qquad \sum d_j \le d$

$$R(z) = \sum_{j=1}^{s} \sum_{\ell=1}^{d_j} \frac{a_{j,\ell}}{(z-t_j)^\ell}, \quad a_{j,d_j-1} \neq 0$$

(unique decomposition)

$$\boxed{\begin{aligned} &\frac{a_0}{z-t} + \frac{a_1}{(z-t)^2} = \sum_{k=0}^{\infty} \frac{m_k}{z^{k+1}} \\ &m_k = a_0 t^k + a_1 \cdot k \cdot t^{k-1} \end{aligned}}$$

$\Rightarrow$

$$(***) \quad \boxed{m_k = \sum_{j=1}^{s} \sum_{\ell=0}^{d_j-1} a_{j,\ell} (k)_\ell \, t_j^{k-\ell}, \quad \sum d_j \le d, \quad k = 0, 1, \dots, 2d-1}$$

$(k)_\ell = k(k-1) \cdot \ldots \cdot (k-\ell+1)$

> Theorem: Let $\operatorname{rank} \tilde{H}_d = r$.
> Problem $(***)$ is uniquely solvable iff $|H_r| \neq 0$.

Recurrence relation:

$$\prod_{j=1}^{s} (E - t_j I)^{d_j} \{m_k\} = 0$$

„discrete difference calculus"

$E\, g(k) = g(k+1)$

$I\, g(k) = g(k)$

Prony's method:

(system II)

1) $[H_d] q = -\begin{bmatrix} m_d \\ \vdots \\ m_{2d-1} \end{bmatrix}$

2) Find roots of q with multiplicities

3) Substitute into (~~★★★~~)

Hankel factorization

$$u_{j,k} = [t_j^k \quad k t_j^{k-1} \quad \dots \quad (k)_{d_j-1} t_j^{k-d_j+1}]$$

$$\mathcal{U} = \begin{bmatrix} u_{1,0} & u_{2,0} & \dots & u_{s,0} \\ \vdots & & & \vdots \\ u_{1,d-1} & & & u_{s,d-1} \end{bmatrix} \quad d \times d$$

"Confluent" Vandermonde matrix

$$\det \mathcal{U} = \prod_{i \neq j} (t_i - t_j)^{d_i d_j} \cdot C_{ij}$$

$$H_d = U B U^T, \quad B = \operatorname{diag}_{j=1..s} B_j,$$

$$B_j := \begin{bmatrix} a_{j,0} & a_{j,1} & & a_{j,d_j-1} \\ a_{j,1} & \iddots & \iddots & \\ \iddots & \binom{d_j-1}{2} a_{j,d_j-1} & & \\ a_{j,d_j-1} & & & \end{bmatrix} \quad a_{j,d_j-1} \neq 0.$$

Stability

„Local approach"

Inverse function theorem:

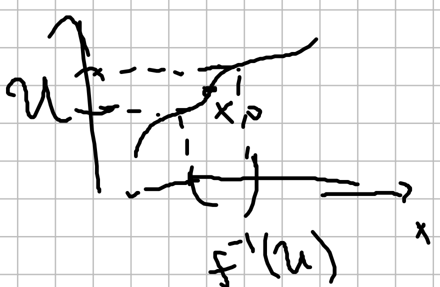
$$f: \mathbb{R} \to \mathbb{R}, \quad f \in C^1$$

x_0 is non-denerate point, $f'(x_0) \neq 0$

$\Rightarrow f$ is locally invertible at x_0.

Let L be Lipschitz constant of f^{-1},

$$f(x_0) = y_0, \quad |f^{-1}(y) - f^{-1}(y_0)| \leq L|y - y_0|$$



$$L = \sup_{x \in f^{-1}(u)} \left| \frac{1}{f'(x)} \right|$$

(Quantitative inverse function theorem).

If f is a differential map $f: \mathbb{C}^n \to \mathbb{C}^n$

we require $f = (f_1, \dots f_n)$, $f_i = f_i(z_1, \dots z_n)$

$$J_f(x_0) = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \dots & \frac{\partial f_1}{\partial z_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial z_1} & & \frac{\partial f_n}{\partial z_n} \end{bmatrix}.$$

If $\det J_f(x_0) \neq 0$, then f is locally invertible
$y_0 = f(x_0)$

$$J_{f^{-1}}(y_0) = \left(J_f(x_0)\right)^{-1}.$$

Taylor series: $f^{-1}(y_0+h) - x_0 \approx \left[J_f(x_0)\right]^{-1} \cdot h$

If $h = \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}$, $|\varepsilon_j| \leq \varepsilon$, then $\varepsilon \ll 1$

we need to estimate $\sum_{j=1}^{n} |\alpha_{ij}| = A_i$, $i = 1, \dots, n$

$\left(J_f(x_0)\right)^{-1} = (\alpha_{ij})$

$\Rightarrow |z_i - \tilde{z}_i| \leq A_i \varepsilon.$

Now we study the „Prony" (Sylvester-Ramanujan).
map

$f = (f_0 \dots f_{2d-1})$, $f_k = m_k = \sum_{j=1}^{d} c_j z_j^k$, $k = 0, \dots, 2d-1.$

$x_0 \in \mathbb{C}^{2d} = (c_1 \dots c_d, z_1 \dots, z_d)$ is a regular pt. of f.

$$\underline{m} = \underline{f} = f(x_0) = V(z_1 .. z_d)\, \underline{c} = [\eta(z_1) \; .. \; \eta(z_d)] \cdot \underline{c}$$

$$\underline{c} = (c_1 .., c_d)$$

$$\eta(z) = \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{2d-1} \end{bmatrix}$$

$$\frac{\partial \underline{m}}{\partial z_j} = c_j \frac{d}{dz} \eta(z) \Big|_{z = z_j} = c_j \eta'(z_j)$$

$$\frac{\partial \underline{m}}{\partial c_j} = \eta(z_j)$$

$$f^{-1} = \begin{bmatrix} \frac{\partial c_1}{\partial m_0} & \cdots & \frac{\partial c_1}{\partial m_{2d-1}} \\ \frac{\partial z_1}{\partial m_0} & & \\ \vdots & & \end{bmatrix}$$

$$J_f = \begin{array}{c} \begin{matrix} \partial c_1 & \partial z_1 & & & \end{matrix} \\ \begin{bmatrix} \eta(z_1) & c_1 \eta'(z_1) & \cdots & \eta(z_d) & c_d \eta'(z_d) \end{bmatrix} \end{array}$$

$$= \underbrace{\begin{bmatrix} \eta(z_1) & \eta'(z_1) & .. & \eta(z_d) & \eta'(z_d) \end{bmatrix}}_{U(z_1 .. z_d)} \begin{bmatrix} 1 & & & & & 0 \\ & c_1 & & & & \\ & & 1 & & & \\ & & & c_2 & & \\ & & & & \ddots & \\ 0 & & & & 1 & \\ & & & & & c_d \end{bmatrix}$$

$$J_f^{-1} = \begin{bmatrix} 1 & & & & \\ & c_1^{-1} & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & c_d^{-1} \end{bmatrix} U^{-1}.$$

So we need to estimate row norms of U^{-1}.

U, U^{-1} investigated by Gautschi, 60's-70's

Lagrange & Hermite Interpolation

$$V^T = \begin{bmatrix} 1 & z_1 & \cdots & z_1^{d-1} \\ \vdots & & & \\ 1 & z_d & \cdots & z_d^{d-1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{d-1} \end{bmatrix} = p_0 + p_1 z_1 + \cdots + p_{d-1} z^{d-1}$$

Let $p(z) = p_0 + p_1 z + \dots + p_{d-1} z^{d-1}$

$$V^T p = \begin{bmatrix} p(z_1) \\ \vdots \\ p(z_d) \end{bmatrix}$$

Interpolation $\Rightarrow$ $p = (V^T)^{-1} \begin{bmatrix} p(z_1) \\ \vdots \\ p(z_d) \end{bmatrix}$ (1)

Lagrange:

(Kronecker delta)

$l_i(z)$ satisfy $l_i(z_j) = \delta_{ij}$.

Substitute into (1) $\Rightarrow$ $\underset{\sim}{l_i} = i^{th}$ column of $(V^T)^{-1}$

coefficients of l_i (in the monomial basis)

$= i^{th}$ row of V^{-1}.

Of course it is well-known

$$l_i(z) = c_i \prod_{j \neq i} (z - z_j) \qquad c_i = \frac{1}{\prod_{j \neq i} (z_i - z_j)}$$

What about $\mathcal{U}$? $\Rightarrow$ Hermite interpolation.

$$\mathcal{U}^T p = \begin{bmatrix} p(z_1) \\ p'(z_1) \\ \vdots \\ p(z_d) \\ p'(z_d) \end{bmatrix}$$

Birkhoff interpolation

Fundamental polynomials

$$\begin{cases} h_i(z_j) = \delta_{ij} \\ h_i'(z_j) = 0 \end{cases}$$

$h \to c$

$g \to z$

$$\begin{cases} g_i(z_j) = 0 \\ g_i'(z_j) = \delta_{ij} \end{cases}$$

[Then the solution to the interpolation problem is

$$p(z) = \sum_{i=1}^{d} p(z_i) h_i(z) + \sum_{i=1}^{d} p'(z_i) g_i(z)$$].

Solution:
$$\begin{cases} h_i(z) = r_i(z)\, \ell_i^2(z) \\ g_i(z) = s_i(z)\, \ell_i^2(z) \end{cases}, \quad \text{where}$$

$$\begin{cases} s_i(z) = z - z_i \\ r_i(z) = 1 - 2\ell_i'(z_i)(z - z_i) \end{cases}$$

$\Rightarrow$ we know all the zeros of h_i, g_i.

Lemma (Gautschi, 1963)

Let $p(z) = \prod_{i=1}^{d} (z - z_i) = \sum_{j=0}^{d} p_j z^j$.

Then
$$\sum_{j=0}^{d} |p_j| \leq \prod_{i=1}^{d} (1 + |z_i|).$$

$\sigma_0 = 1$

Pf: Let $\sigma_m(z_1, \ldots, z_d) = \sum_{1 \leq i_1 < i_2 < \ldots < i_m \leq d} z_{i_1} \cdots z_{i_m}$, $m \leq d$

(elementary symmetric polynomial)

$\underline{d=3}$

$\sigma_1 = z_1 + z_2 + z_3$

$\sigma_2 = z_1 z_2 + z_2 z_3 + z_1 z_3$

$\sigma_3 = z_1 z_2 z_3$

$$p(z) = \sum_{j=0}^{d} (-1)^j \sigma_j z^{d-j}$$

$$\Rightarrow \sum_{j=0}^{d} |p_j| = \sum_{j=0}^{d} |\sigma_j|$$

<u>Case I</u>

Assume $z_j \geq 0$ $\forall j$, then $\sigma_j(z_1, \ldots, z_d) \geq 0$

$$p(-1) = (-1)^d \sum \sigma_j = (-1)^d \prod_{i=1}^{d} (1 + z_i).$$

$$\sum_{j=0}^{d} \sigma_j(z_1, \ldots, z_d) = \prod_{i=1}^{d} (1 + z_i)$$

<u>Case II</u> z_j arbitrary

$$\Rightarrow |\sigma_j(z_1, \ldots, z_d)| \leq \sigma_j(|z_1|, \ldots, |z_d|)$$

$$\Rightarrow \sum_{j=0}^{d} |p_j| \le \sum_{j=0}^{d} \sigma_j(|z_1|, \dots, |z_d|) = \prod_{i=1}^{d} (1 + |z_i|).$$

Now apply the lemma to $\{g_i, h_i\}$.

For simplicity, $|z_i - z_j| \approx \Delta \ll 1$, $|z_j| = 1$

$$\therefore\ c_i \approx \Delta^{1-d} \Rightarrow \sum_{j=0}^{d} |g_{i,j}| \approx \Delta^{2-2d} \le c_i^2 \cdot 2^d$$

$$l_i'(z_i) = \sum_{j \ne i} \frac{1}{z_i - z_j} \approx \frac{1}{\Delta}$$

$$\Rightarrow \sum_{j=0}^{d} |h_{i,j}| \approx \Delta^{1-2d}.$$

Finally

$$\left\{ \begin{aligned} &|c_i - \tilde{c}_i| \approx \Delta^{1-2d} \cdot \varepsilon \\ &|z_i - \tilde{z}_i| \approx \frac{\Delta^{2-2d}}{|c_i|} \cdot \varepsilon \end{aligned} \right. \Bigg\|,$$

Next:

1) $m(k) = \sum_{j=1}^{d} c_j e^{i t_j k}$ $k = 0, .., N$

$N >> 2d-1$

over determined case.

2) We will want to obtain quantitive estimates

$\boxed{\varepsilon << 1}$ $\varepsilon < \varepsilon(N, \Delta, d)$.