Sparse SR

$$\sum_{j=1}^{d} c_j \delta(t-t_j) \xrightarrow{\mathcal{F}} \sum_{j=1}^{d} c_j e^{it_j\omega} \xrightarrow[\text{sample}]{\text{unif.}} \underbrace{\sum_{j=1}^{d} c_j z_j^k}_{\text{measurement model}} \quad z_j = e^{it_j}$$

$\omega = k = 0, 1, 2, \ldots$

$(2d \cdot \Delta\omega) \Leftarrow \omega = \Delta\omega \cdot k \quad \ldots$

- "Imaging of point sources"
- Line Spectrum estimation $\sum_{j=1}^{d} c_j \delta(\omega-\omega_j) \to \sum_{j=1}^{d} c_j e^{i\omega_j t}$
- Sparse Fourier transform
- Hidden periodicities
- Sparse representations | compressed sensing &.
- Moment problems (Classical) $\quad \mu$ - Borel measure

  Given: $\int x^k \, d\mu = m_k$

  $\{m_0, m_1, \ldots\} \to$ recover μ.

Problem: Given $\{m_0, .. m_{2d-1}\}$, find $\{c_j, z_j\}_{j=1}^{d}$ s.t.

$$(*) \qquad m_k = \sum_{j=1}^{d} c_j z_j^k \quad , \qquad k=0,1,..,2d-1$$

find $c_1, z_1, .., c_d, z_d$ s.t

$$2d \begin{cases} c_1 + c_2 + .. + c_d = m_0 \\ c_1 z_1 + c_2 z_2 + .. + c_d z_d = m_1 \\ \vdots \\ c_1 z_1^{2d-1} + \dots c_d z_d^{2d-1} = m_{2d-1} \end{cases}$$

I) Existence
II) Uniqueness

Uniqueness

Theorem 1 the solution, if exists, is unique, up to permutations

We associate $\{c_j, z_j\}_{j=1}^{d}$ with point measure

$$\mu = \sum_{j=1}^{d} c_j \delta(z - z_j)$$

$(*)$ is equivalent to the moment problem:

Find μ, with $|\text{supp}\, \mu| \le d$

s.t. $\int z^k d\mu = m_k, \quad k=0,1,.. 2d-1$

Suppose μ_1, μ_2 solutions to $(*)$

$\rho = \mu_1 - \mu_2$ is a point measure, $|\operatorname{supp} \rho| \leq 2d$

s.t. $\int z^k \, d\rho = 0$, $k = 0, 1, \dots 2d-1$.

Let $p = |\operatorname{supp} \rho|$

- if $p = 0 \Rightarrow$ we are done
- else, $\rho = \sum_{j=1}^{p} b_j \delta(z - \omega_j)$

$$V(\omega_1, \dots \omega_p) = \begin{bmatrix} 1 & \dots & 1 \\ \omega_1 & & \omega_p \\ \omega_1^2 & & \omega_p^2 \\ \vdots & & \vdots \\ \omega_1^{p-1} & & \omega_p^{p-1} \end{bmatrix} \in \mathbb{C}^{p \times p}$$

Vandermonde matrix

Known: $|V| = \prod_{1 \leq i < j \leq p} (\omega_j - \omega_i) \leftarrow$ (induction),

$$\sum_{j=1}^{p} b_j \omega_j^k = 0 \qquad k = 0, 1, .., 2p-1 \qquad p \leq d$$

$$V \cdot \underline{b} = 0 \Rightarrow \underline{b} = 0 \Rightarrow \rho = 0.$$

$\underline{b} = (b_1, .., b_p)$

Existence.

Example: $m_0 = m_1 = \dots m_{d-1} = 0$ but $m_k \neq 0$ for some $d \le k \le 2d-1$

Example: $m_k = a \cdot z^k = z \cdot m_{k-1}$ $\forall k \geq 0$.

$d=1$

$$z = \frac{m_k}{m_{k-1}}$$

Example: $d=2$

$$\left.\begin{aligned} m_k &= az^k + bw^k \\ m_{k+1} &= az^{k+1} + bw^{k+1} \\ m_{k+2} &= az^{k+2} + bw^{k+2} \end{aligned}\right\} \Rightarrow \boxed{zw\, m_k - (z+w)\, m_{k+1} + m_{k+2} = 0}$$

$zw \quad -(z+w) \quad 1$

$(x-z)(x-w)$

<u>Claim 1</u> Let μ be a solution, $\mu = \sum_{j=1}^{d} c_j \delta(z - z_j)$

Define $P(z) = \prod_{j=1}^{d} (z - z_j) = p_0 + p_1 z + \dots + p_{d-1} z^{d-1} + p_d z^d$,

$p_d = 1$

Then: $\sum_{j=0}^{d} p_j m_{k+j} = 0$.

$\forall k \geq 0$

Pf. $\sum_{j=0}^{d} p_j m_{k+j} = \sum_{j=0}^{d} p_j \sum_{i=1}^{d} c_i z_i^{k+j} = \sum_{i=1}^{d} c_i z_i^k \sum_{j=0}^{d} p_j z_i^j$ (**)

$= \sum c_i z_i^k P(z_i) = 0$.

Write this as a homogeneous system

$$\begin{array}{cc} & d+1 \\ d & \begin{bmatrix} m_0 & m_1 & \cdots & m_d \\ m_1 & m_2 & \cdots & m_{d+1} \\ m_2 & \cdots & & \\ \vdots & & & \\ m_{d-1} & \cdots & & m_{2d-1} \end{bmatrix} \end{array} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_d \end{pmatrix} = 0 \qquad (1).$$

Claim 2 If μ is a solution, and $(q_0 \ldots q_d)$ is a non-trivial solution of (1),

and $Q(z) = q_0 + q_1 z + \ldots + q_d z^d$

then $Q(z_i) = 0 \quad \forall z_i \in \operatorname{supp} \mu.$

Pf: by (*) $\displaystyle\sum_{i=1}^{d} c_i z_i^k Q(z_i) = \sum_{j=0}^{d} q_j m_{k+j} = 0 \qquad k = 0, 1, \ldots d-1$

$$\rho = \sum_{i=1}^{d} c_i Q(z_i) \delta(z - z_i) \Rightarrow \int z^k d\rho = 0, \quad k = 0, 1, \ldots d-1$$

$\Rightarrow$ we have already seen that $\rho = 0$

$\Rightarrow Q(z_i) = 0 \quad \checkmark$

Def: μ is called a regular solution to (*) if $|\operatorname{supp} \mu| = d$.

Let μ be a solution, $\mu = \sum_{j=1}^{d} c_j \delta(z - z_j)$

$$H_d := \begin{bmatrix} m_0 & \cdots & m_{d-1} \\ & & \\ m_{d-1} & \cdots & m_{2d-2} \end{bmatrix}$$

<u>Claim 3</u>: $H_d = \underset{V(z_1 .. z_d)}{V} \cdot \begin{bmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_d \end{bmatrix} V^T$ pf by inspection

Corollary: if μ is a solution then
$|\operatorname{supp} \mu| = \operatorname{rank} H_d$.

Suppose $\operatorname{rank} H_d = d$. Then $\exists$ unique solution to

$$\begin{bmatrix} H_d & \begin{matrix} m_d \\ \vdots \\ m_{2d-1} \end{matrix} \end{bmatrix} \begin{bmatrix} q_0 \\ \vdots \\ q_{d-1} \\ 1 \end{bmatrix} = 0 \implies H_d \begin{bmatrix} q_0 \\ \vdots \\ q_{d-1} \end{bmatrix} = - \begin{bmatrix} m_d \\ \vdots \\ m_{2d-1} \end{bmatrix}$$

$$Q(z) = z^d + q_{d-1} z^{d-1} + \ldots + q_0$$

Assume $Q(z)$ does not have multiple roots.

$\{w_1 .. w_d\}$

$$V(w_1,..,w_d)\cdot\tilde{c} = \begin{pmatrix} m_0 \\ \vdots \\ m_{d-1}\end{pmatrix} \to \text{find } \tilde{c}.$$

$$\tilde{\mu} = \sum_{j=1}^{d} \tilde{c}_j\,\delta(z-w_j)$$

Need to prove that $\int z^k\, d\tilde{\mu} = m_k$, $k=d, d+1, .. 2d-1$.

(already shown for $k=0,..\,d-1$).

Define $n_k = \sum \tilde{c}_j w_j^k$, $k=0,1,.., 2d-1$

Need to show: $n_k = m_k$, $k=d,.., 2d-1$.

$\tilde{\mu}$ is a solution to $(*)$ with data $\{n_0 .. n_{2d-1}\}$.

$$\Rightarrow \begin{cases} \sum_{j=0}^{d} n_{k+j}\, q_j = 0, & k=0,1,..\,d-1. \\ \sum_{j=0}^{d} m_{k+j}\, q_j = 0, & k=0,1,..,d-1 \end{cases}$$

but also — by construction.

Consider $\sum_{j=0}^{d} a_{k+j}\, q_j = 0$, $k=0,1,..$

$q_d = 1$.

$\Rightarrow$ Given „initial condition" $\{a_0,.., a_{d-1}\}$ the solution is determined uniquely $\Rightarrow m_k = n_k$ $\forall k$.

$\boxed{\begin{array}{l}\text{Th 2: } (*) \text{ has a regular solution iff} \\ \qquad |H_d| \neq 0 \\ \text{and} \qquad Q(z) \text{ has no multiple roots.}\end{array}}$

What about singular solutions

Suppose μ is a solution with $|\text{supp}\,\mu| = p < d$.

$\Rightarrow$ rank $H_d = p$.

But of course μ solves $(*)_p$

$(*)$ with $2p-1$ equations

$\Rightarrow |H_p| \neq 0$, $Q_p(z)$ has no multiple roots.

<u>Theorem 3</u> This is also sufficient, i.e.

rank $H_d = p$.

$|H_p| \neq 0$, $Q_p(z)$ has no multiple roots $\Rightarrow$

$\exists$ a solution μ to $(*)_d$ with $|\text{supp}\,\mu| = p$.

Pf: Let μ be a solution to $(*)_p$ (by Th 2).

$\Rightarrow \int z^k \, d\mu = m_k$, $k = 0, 1, \ldots, 2p-1$.

$$\begin{bmatrix} m_0 & \cdots & m_p & \cdots & m_d \\ \vdots & H_p & \vdots & & \vdots \\ m_{p-1} & \cdots & m_{2p-1} & & \vdots \\ \vdots & & & & \vdots \\ m_{d-1} & \cdots & & & m_{2d-1} \end{bmatrix} \begin{bmatrix} q_0 \\ \vdots \\ q_{p-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0$$

$\{n_k\}$ - moments of μ. $n_k = m_k$

Complete this (exercise).

$\{m_0 \ldots m_{2d-1}\}$

"Unsolvability set" is a set of measure zero in $\mathbb{C}^{2d}$ . .

- Noiseless SR problem can achieve "infinite resolution".

Prony's method. 1795 $\sum c_j e^{\lambda_j t}$

Sylvester
Ramanujan } Sylvester-Ramanujan

Padé approximation

Define
$$R(z) = \sum_{j=1}^{d} \frac{a_j}{1 - z z_j} = \sum_{j=1}^{d} a_j \sum_{k=0}^{\infty} (z z_j)^k$$

$$\left(\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \right)$$

$$= \sum_{k=0}^{\infty} m_k z^k .$$

(moment-generating function).

Convergence for $z \gg 1$

$$R(z) = \sum_{j=1}^{d} \frac{a_j}{z - z_j} = \sum_{k=0}^{\infty} \frac{m_k}{z^{k+1}} = \frac{P(z)}{Q(z)}$$

From $2d$ Taylor coefficients we would like to get an appr. $\approx \frac{1}{z^{2d}}$.

$$\begin{cases} \deg P \leq d-1 \\ \deg Q \leq d \end{cases}.$$

$$P(z) = b_0 + \dots + b_{d-1} z^{d-1} \qquad d$$
$$Q(z) = c_0 + \dots + c_d z^d \qquad d+1$$

2d independent coeffs

$\Rightarrow$ Solving (*) is reduced to finding a rational f-n whose first $2d$ Taylor coeffs at $z = \infty$ are given.