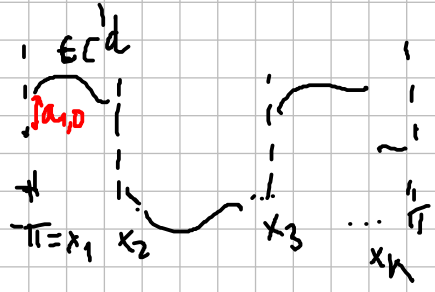
- Algebraic Fourier reconstruction of piecewise-smooth functions
- Complete Algebraic reconstruction of — " — " — " — from Fourier data.

Given: $\{c_k(f)\}_{|k|=0}^{N}$. $N \gg 1$. (N^{-d})

Model: f piecewise-smooth

- $f_j \in C^d$
- $\{x_j\}_{j=1}^{K}$ - discont. points
- $\{a_{\ell,j}\}_{j=1..K}^{\ell=0,\dots,d}$



$0 < m < |a_{0,j}|$ $\quad\quad \min_{j \neq \ell} |x_j - x_\ell| = \Delta > 0$

$$f = \underbrace{\Phi_d}_{\text{piecewise polynomial}} + \psi^{\in C^d}$$

$$\underset{\text{given}}{c_k(f)} = \underset{\frac{1}{k}}{c_k(\Phi_d)} + \underset{\sim k^{-(d+2)}.}{c_k(\psi)} \quad |k| < N$$

perturbation term

$$|c_k(\psi)| \le R \cdot k^{-(d+2)}$$

$$(\|\psi\|_2)$$

$\left(|\tilde{x}_j - x_j| \sim N^{-\delta/4} \right)$

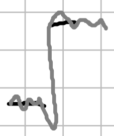
I) Approximate $\{x_j\}$ up to $1/N$ want

II) 

III) $c_k(\Phi_{d_j}) \to \{x_j, \{a_{\ell,j}\}_{\ell=0}^{d}\}$ $\forall j$

E. Tadmor Filters, mollifiers & computation of the Gibbs phenomenon 2007

Step I



$S_N(f)$

thresholding of $(S_N f)'$

$$D_N' * f \approx (S_N f)'(x) \approx \begin{cases} a_{0,j} + O\left(\frac{\log N}{N}\right) & x \approx x_j \\ O\left(\frac{\log}{N}\right) & |x - x_j| \gg 1/N \end{cases}$$

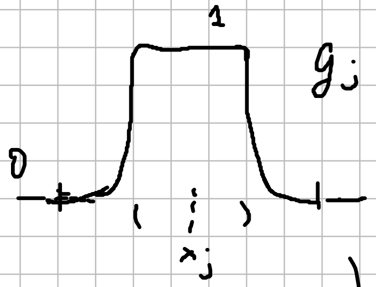
highpass spectral filter $\sigma(t) \in [0,1]$ $\quad \frac{\sigma(s)}{s} \in C^2[0,1]$

$$K_\sigma(t) = \sum_{|k| \le N} \sigma\left(\frac{|k|}{N}\right) e^{ikt}$$

$\to$ Threshold $K_\sigma * f$

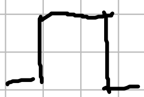
Step II

-1



$g_j \in C^\infty$

$$g_j = \begin{cases} e^{-\frac{1}{1-|x|^2}} & |x|<1 \\ 0 & |x|\geq 1 \end{cases} \in C^\infty$$

$g_j \, *$ 

$\to$ compute $\overset{M_k}{c_k(g_j \cdot f)} = (\hat{g}_j * \hat{f})_k$

$\to$ want to control the error $|M_k - M_k'| \leq k^{-(d+2)}$

$$c_k(g_j) \lesssim k^{-\alpha} \quad \forall \alpha > 0$$

III) $$(\bar{\Phi}_d)^{(d+1)} = \frac{1}{2\pi}\sum_{j,l} b_{l,j}\, \delta^{(l)}(x - x_j)$$

$\{a_{0,j}\} \to b_{d,j}$

$b_{l,j} = a_{d-l,j}$

$$c_k(\Phi_d) = \frac{1}{2\pi}\sum_{j=1}^{n} e^{-ikx_j}\sum_{l=0}^{d}\frac{(-1)^l b_{l,j}}{(ik)^{l+1}}$$

For each j

$w = e^{-ix_j}$

$$\tilde{c}_k = \frac{w^k}{2\pi}\sum_{l=0}^{d}\frac{b_l}{(ik)^{l+1}} + \varepsilon_k, \qquad |k| = 0, \dots, N$$

$|\varepsilon_k| \lesssim k^{-d-2}$

$$\tilde{m}_k = 2\pi\, \tilde{c}_k (ik)^{d+1} = w^k \sum_{l=0}^{d} b^*_{d-l}\, k^l + O(k^{-1})$$

$$b^*_{d,l} = \frac{b_l}{i^{l+1}}$$

$$m_k := \omega^k \sum_{\ell=0}^{d} b^*_{d-\ell}\, k^\ell \qquad |k| = 0, \dots, N$$

$d+2$ unknowns

Elimination of $\{b^*_{d-\ell}\}$

$$n = \left\lfloor \frac{N}{d+2} \right\rfloor \qquad z = \omega^n \in \text{unit circle}$$

$$f_j = m_{jn} = z^j \sum_{\ell=0}^{d} b^*_{d-\ell}\, j^\ell n^\ell \qquad \left(\begin{array}{c}\text{polynomial-}\\ \text{exponential}\\ \text{sum}\end{array}\right)$$

recurrence relation for $\{f_j\}$

$$P_n(u) = \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} f_{j+1}\, u^{d+1-j}$$

Fact: z is a root of $P_n(u)$.

$$\left[\text{Hint: } \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} \underbrace{(j+1)^\ell}_{\psi(j)} = \Delta^{d+1}\psi = 0\right]$$

finite diff. operator
$$\left((\Delta\psi)(j) = \psi(j+1) - \psi(j)\right)$$

$\rightsquigarrow$ Compute $\tilde{p}_n$ from $\tilde{m}_k$ and find $\tilde{z}$ which is closest to the unit circle.

Q: $|\tilde{z} - z| \le ?$

Rouché's theorem:

$$q(z) = p(z) + e(z)$$

Assume: z_0 a simple zero of p

If $\exists \rho \in \mathbb{R}^+$ s.t. $|p(z)| > |e(z)|$ for $|z - z_0| = \rho$

then $q(z)$ has a simple zero in $B_\rho(z_0)$

$\rightarrow$ want upper bound on ρ

$p_n \qquad \tilde{p}_n$

$\downarrow$

$$s_\ell(t) = \sum_{j=0}^{d} (-1)^j \binom{d+1}{j} (j+1)^{\ell} t^{\ell}$$

$t \in \mathbb{R}$ $$p_n(zt) = z^{d+2} \sum_{\ell=0}^{d} b^*_{d-\ell}\, n^{\ell}\, s_\ell(t)$$

$\rightsquigarrow n \gg 1$ $\quad p_n$ is essentially s_d

„Hard analysis": s_d has simple zeros $[1, \infty)$

zeros of $p_n \rightarrow$ zeros of s_d

$\tilde{p}_n$ vs $p_n \rightarrow$ Rouché's theorem again

$\rho \approx n^{-d-1}$ is OK.

$\Rightarrow$ Result: $\boxed{|\hat{z} - z| \approx N^{-d-1}}$