Lecture 4 

# Fourier analysis

$f(x) : [0,2\pi] \to \mathbb{R}$ (or $\mathbb{C}$)

$$c_n := \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} f(x)\,dx \quad n^{\text{th}} \text{ Fourier coefficient}$$

Synthesis: $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$

Smoothness vs. decay

- jumps - $c_n \sim 1/n$
- $f \in C^0 \to c_n \sim 1/n^2$ (e.g. corners)
- $f \in C^d \Rightarrow c_n \sim n^{-d-2}$
- analytic $\Rightarrow c_n \sim r^n,\ r<1$

Riemann-Lebesgue: $f \in L^1 \Rightarrow a_k \to 0$.

Convergence: somewhat subtle

Global: $\begin{cases} f\in L^1 \Rightarrow \text{can diverge everywhere} \\ f \in L^2 \Rightarrow \text{converges a.e. in mean sq.} \\ f \in C^\alpha \Rightarrow \text{uniform convergence everywhere} \\ f\in BV \Rightarrow \text{convergence everywhere} \end{cases}$

Local: $\begin{cases} f \text{ differentiable at } x \Rightarrow \text{converges at } x \\ f \text{ has a finite jump} \Rightarrow \text{conv. to midpoint} \\ \text{(Dini condition ...)} \end{cases}$

---

Dirichlet kernel: truncate "$\delta(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{inx}$"

$$\delta_N(x) = \frac{1}{2\pi}\sum_{n=-N}^{N} e^{inx} = \frac{1}{2\pi}\frac{\sin(N+1/2)x}{\sin x/2}$$

Weak convergence: $\int \delta_N(x) f(x)\,dx = a_0 + \dots + a_N \to f(0)$.

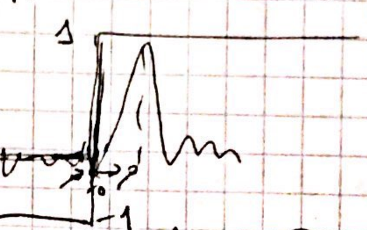
Gibbs oscillations



$$f \sim \sum c_k e^{ikx}$$

$$f_N(x) = \sum_{k=-N}^{N} c_k e^{ikx} = \frac{1}{2\pi}\int_0^{2\pi} f(t) e^{-ikt}\,dt\; e^{ikx} = \int_0^{2\pi} \delta_N(y) f(x-y)\,dy$$
$$= (\delta_N * f)(x)$$

What happens near the jump?




SW (square wave)

Choose $x_0$ for which $\delta_N(x_0) = 0 \to x_0 = \frac{\pi}{N+1/2}$

As $N \to \infty$, sidelobes contribute $\to 0$ (cancel)

From main lobe $\to J = \int_{-\pi}^{\pi} SW(t)\,\delta_N(x_0 - t)\,dt \to \int_{-x_0}^{x_0} \delta_N(t)\,dt$

Change of var: $y = \frac{N+1/2}{\pi} t$

$$J \approx \frac{\pi}{N+1/2}\frac{1}{2\pi}\int_{-1}^{1} \frac{\sin \pi y}{\sin \frac{\pi y}{2(N+1/2)}}\,dy \approx \int_{-1}^{1} \frac{\sin \pi y}{\pi y}\,dy \approx 1.17898.$$

---

Convolution: $f, g$ periodic, $f \sim \sum \hat{f}(k) e^{ikx}$, $g \sim \sum \hat{g}(k) e^{ikx}$

$$h(x) = \int f(t) g(x-t)\,dt = \ldots = 2\pi \sum \hat{f}(k)\hat{g}(k) e^{ikx}$$

$$\Rightarrow \hat{h}(k) = 2\pi \hat{f}(k)\hat{g}(k)$$

Convolution of sequences: $(c*d)_n = \sum_{k=-\infty}^{\infty} c_k d_{n-k} \Rightarrow f(x)g(x) = \sum_{n=-\infty}^{\infty} (c*d)_n e^{inx}$.

Derivative: $\left(\sum c_k e^{ikx}\right)' = \sum (ik) c_k e^{ikx}$

Integrals: $\int \left(\sum c_k e^{ikx}\right) dx = \sum_{k\neq 0} \frac{c_k}{ik} e^{ikx}$ $\quad (k \neq 0)$.

# Discrete signals

"Dual" Fourier series

$\{x[n]\}_{n=-\infty}^{\infty}$, e.g. samples of a continuous-time signal.

Discrete-Time Fourier Transform

$$X_{2\pi}(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-in\omega}$$

→ continuous fn of frequency, $2\pi$-periodic.

Exists if $x[n]$ is absolutely summable.

Inversion formula:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_{2\pi}(\omega)\, e^{in\omega}\, d\omega$$

Crash-course on DSP

$$x = \sum x[k]\,\delta[n-k] \qquad \delta = \text{unit impulse}$$

Linear system: (filter)

$$\begin{cases} y[n] = (Tx)[n] = \sum_k y[k]\,\delta[n-k] = \sum_k x[k]\, T\{\delta[n-k]\} \\ T\{\delta[n-k]\} \equiv h_k[n] \end{cases}$$

$$\Rightarrow (Tx)[n] = (x * h)_n$$

Time-invariant system: $h_k[n] = h_0[n-k]$

Impulse response

Switching between time & frequency in convolution thm.:

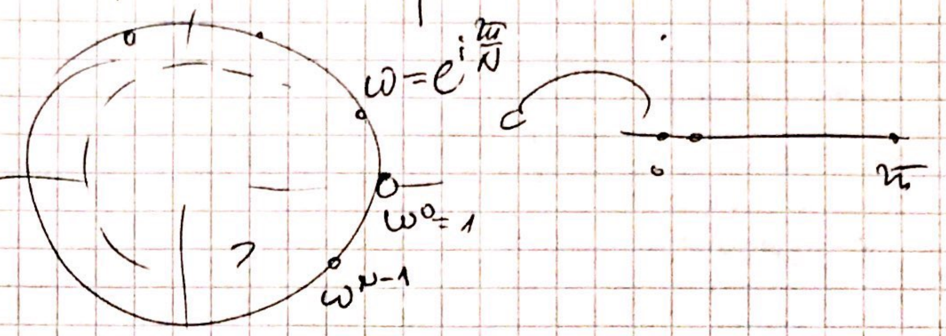
$$\sum_{n=-\infty}^{\infty} (x * h)_n\, e^{-in\omega} = X_{2\pi}(\omega)\, H_{2\pi}(\omega)$$

frequency response

convolution (time/freq) ⟺ multiplication (freq/time)

# Discrete Fourier Transform

Divide $[0, 2\pi]$ into $N$ equal intervals




Think about $f \in \mathbb{C}^N$ as samples of $f(x)$ on the grid $\left\{ k \frac{2\pi}{N} \right\}_{k=0}^{N-1}$

DFT basis: $\left\{ e^{ik\bar{x}} \right\}_{k=0,\dots,N-1}$ $\quad \bar{x} = \{x_j\}_{j=0}^{N-1} = \left\{ j\frac{2\pi}{N} \right\}_{j=0}^{N-1}$

approximate $c_k(f) = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-ikx}\,dx$ by

$$\hat{f}_N = \frac{1}{2\pi}\sum_{j=0}^{N-1} f\left(j\frac{2\pi}{N}\right) e^{-ikj\frac{2\pi}{N}} \cdot \frac{2\pi}{N} = \frac{1}{N}\sum_{j=0}^{N-1} f\left(\frac{j}{N}2\pi\right)\omega^{-jk}$$

$$\hat{f}_N = \frac{1}{N} F_N^* \cdot f_N \quad , \quad F_N = \begin{bmatrix} \omega^0 & \omega^0 & \cdots & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^{N-1} \\ \vdots & \vdots & \vdots & \vdots \\ \omega^0 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{(N-1)^2} \end{bmatrix}, \quad \omega^N = 1.$$

$$F_N^* F_N = N \cdot I_N$$

(Matlab fft = multiplication by $F_N^*$).

Fast algorithm (FFT) ⇒ perform 2 DFT's on even/odd then combine

Complexity $\sim \frac{1}{2} N \log_2 N$

$$F_N^* = \begin{bmatrix} I_M & D_M \\ I_M & -D_M \end{bmatrix} \begin{bmatrix} F_M^* & 0 \\ 0 & F_M^* \end{bmatrix} \times \begin{bmatrix} \text{permutation} \\ \text{matrix} \end{bmatrix}$$

# Convolutions & DFT



Cyclic shift $P = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & & 0 \\ 0 & 1 & & \vdots \\ \vdots & \vdots & & \\ 0 & & & 0 \end{bmatrix}$

$x \in \mathbb{C}^N$

Let $X = [x \; Px \; \cdots \; P^{N-1}x] = \begin{bmatrix} x_0 & x_{N-1} & \cdots & x_1 \\ x_1 & x_0 & \cdots & x_2 \\ \vdots & & \ddots & \\ x_{N-1} & x_{N-2} & \cdots & x_0 \end{bmatrix}$ circulant matrix

Let $c = \frac{1}{N} F_N^* x \Rightarrow \left(\frac{1}{N} F_N^* P x\right)_k = \omega^{-k} c_k$

$$\Rightarrow \frac{1}{N} F_N^* X = \begin{bmatrix} c_0 & & \\ & \ddots & \\ & & c_{N-1} \end{bmatrix} F_N^*$$

$$\Rightarrow X = \underbrace{F_N}_{\text{ifft}} \, \underbrace{\text{diag}(c)}_{\text{fft}(x)} \, \underbrace{F_N^*}_{\text{fft}}$$

$d \in \mathbb{C}^N \Rightarrow Xd$ is a circular / cyclic convolution

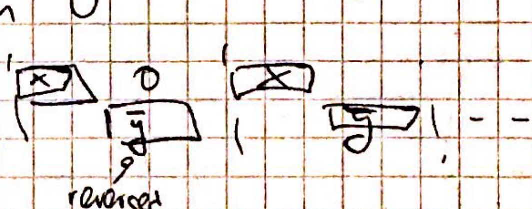
$$(x \circledast d)_n = \sum_{k=0}^{N-1} x_{n-k} d_k, \quad n = 0, \dots, N-1.$$
$$= \sum_{j=0}^{N-1} x_j d_{n-j}$$

Application: filters

If both $x, y$ are finite signals,

want to compute $(x * y)_k = \sum_{n=-\infty}^{\infty} x[n] y[k-n]$

Pad with 0




and apply circular convolution.

$f(x) = \sum c_k e^{ikx} \qquad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-ikt}\,dt$

$2\pi \to T \Rightarrow f(x) = \frac{1}{T}\sum c_k e^{ik\frac{2\pi}{T}x}, \quad c_k = \int_{-T/2}^{T/2} f(t) e^{-i\frac{2\pi}{T}kt}\,dt$

$\xi = \frac{2\pi k}{T}, \quad \text{"} d\xi = \frac{2\pi}{T} \text{"} \Rightarrow c(\xi) = \int_{-T/2}^{T/2} f(t) e^{-i\xi t}\,dt$

$f(x) = \frac{1}{T}\sum c_k e^{ik\frac{2\pi}{T}x} \text{"=" } \frac{1}{2\pi}\sum_k c(\xi) e^{i\xi x}\,d\xi$

$T \to \infty \Rightarrow \begin{cases} \hat{f}(\xi) = \int_{-\infty}^{\infty} f(t) e^{-i\xi t}\,dt \\ f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x}\,d\xi \end{cases}$

Existence: $f \in L^1$.

Convolution: $f * g = \int_{\mathbb{R}} f(x-y) g(y)\,dy$

$(f * g)^\wedge(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.

$\widehat{\partial_x f}(\xi) = i\xi \hat{f}(\xi)$

$\left(\int^x f\right)^\wedge = \frac{1}{i\xi}\hat{f}(\xi)$

$\widehat{f(x-d)} = e^{-i\xi d}\hat{f}(\xi)$

---

$\hat{f} = \chi[-\Omega,\Omega] = \begin{cases} 1 & |\omega| \le \Omega \\ 0 & \text{else} \end{cases}$

$\Rightarrow f(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} e^{i\xi t}\,d\xi = \frac{\sin \Omega t}{\pi t} = \frac{\Omega}{\pi}\operatorname{sinc}\frac{\Omega t}{\pi}$

Super-important $\rightarrow$ function! $\quad \operatorname{sinc}\xi = \frac{\sin(\pi\xi)}{\pi\xi}$.

Poisson summation formula

- Start with function $f(x)$
- Choose period $T$
- Periodization

$$f_T(x) = \sum_{k=-\infty}^{\infty} f(x+kT) \quad \leftarrow \text{periodic}$$

- Fourier series:

$$c_k(f_T) = \frac{1}{T}\int_{-T/2}^{T/2} f_T(x) e^{-i\frac{2\pi}{T}kx} dx$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \sum_{s=-\infty}^{\infty} f(x+sT) e^{-i\frac{2\pi k}{T}x} dx$$

$$= \frac{1}{T}\sum_{s=-\infty}^{\infty} \int_{-T/2}^{T/2} f(x+sT) e^{-ik\frac{2\pi}{T}x} dx$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} f(x) e^{-ik\frac{2\pi}{T}x} dx = \frac{1}{T}\hat{f}\left(\frac{2\pi k}{T}\right).$$

Therefore:

$$\boxed{f_T(x) = \frac{1}{T}\sum_k \hat{f}\left(\frac{2\pi k}{T}\right) e^{i\frac{2\pi}{T}kx}} \qquad (1)$$

Analogously: $\tau$ - sampling period

$$\hat{f}_{2\pi/\tau}(\xi) = \sum_{n=-\infty}^{\infty} \hat{f}\left(\xi + \frac{2\pi}{\tau}n\right) = \tau \sum_k f(\tau k) e^{-i\tau k\xi}$$

Nyquist sampling theorem

Suppose $f$ is band limited, with band limit $\frac{\pi}{\tau}$:

$$\hat{f}(\xi) = \hat{f}_{2\pi/\tau}(\xi)\cdot \chi_{[-\pi/\tau,\, \pi/\tau]}.$$

Then we can recover $f$ from $\{f(\tau n)\}$:

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi) e^{i\xi x} d\xi = \frac{1}{2\pi}\int_{-\pi/\tau}^{\pi/\tau} \hat{f}_{2\pi/\tau}(\xi) e^{i\xi x} d\xi$$

$$= \frac{\tau}{2\pi}\int_{-\pi/\tau}^{\pi/\tau} \sum_k f(\tau k) e^{-i\tau k\xi} e^{i\xi x} d\xi$$

$$= \frac{\tau}{2\pi}\sum_k f(\tau k)\int_{-\pi/\tau}^{\pi/\tau} e^{i\xi(x-\tau k)} d\xi$$

$$= \boxed{\sum_k f(\tau k)\, \mathrm{sinc}\,\frac{x-\tau k}{\tau} = f(x)}$$

Band limited functions $\quad$ $\operatorname{supp}\hat{f} \subset [-\Omega, \Omega]$, $\Omega < \pi/\tau$

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}\hat{f}(\xi) d\xi = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} e^{i\xi x}\int_{\mathbb{R}} e^{-i\xi t} f(t) dt$$

$$= \int_{\mathbb{R}} \frac{\sin \Omega(x-t)}{\pi(x-t)} f(t) dt = \frac{\Omega}{\pi}\int_{\mathbb{R}} f(t)\, \mathrm{sinc}\, \frac{\Omega}{\pi}(x-t)\, dt$$

$$\boxed{\text{Ideal Low-pass filter: } P_\Omega f = \int_{\mathbb{R}} \frac{\sin\Omega(t-t')}{\pi(t-t')} f(t') dt'}$$