Lecture 10

$$\{c_j, z_j\}_{j=1}^{d} \to \left\{ \sum_{j=1}^{d} c_j z_j^k + \varepsilon_k \right\}_{k=0,\dots,2d-1}, \quad |z_j| = 1.$$

$$|c_j - \hat{c}_j| \quad , \quad |z_j - \hat{z}_j|$$

$$P: \mathbb{C}^n \to \mathbb{C}^N$$

Linearization

$$L_x(z) = P(x) + J(x)(z - x)$$

$$x \in \mathbb{C}^n$$

$$y = P(x) + e, \quad \|e\| \leq \varepsilon \ll 1$$

Least squares: solve $\arg\min_z \|y - P(z)\|^2$ $\leftarrow$ hard to analyze

Instead

$$x^* = \arg\min_z \|y - L_x(z)\|^2 = J^+ y$$

$$\Rightarrow x - x^* = J^+ e \quad , \quad J^+ = (J^* J)^{-1} J^*$$

$\Rightarrow$ Need to consider „component-wise" condition numbers

$j = 1 \dots, n$ $$\mathcal{K}_j(x) := \sum_{i=1}^{N} |(J^+(x))_{j,i}|$$

For the Prony map: $\{c_j, z_j\}_{j=1}^{d} \to \left\{\sum_{j=1}^{d} c_j z_j^k\right\}_{k=0}^{N}$

$$J_P = \left[\eta(z_1)\ \ c_1\eta'(z_1) \ \dots\ \eta(z_d)\ \ c_d\,\eta'(z_d)\right]$$

where $\eta(z) = \begin{bmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^N \end{bmatrix}$

$$= \underbrace{\left[\eta(z_1)\ \ \tilde\eta(z_1) \ \dots\ \eta(z_d)\ \ \tilde\eta(z_d)\right]}_{\text{Pascal-Vandermonde matrix}} \times \qquad \tilde{\mathcal{U}}_N = \tilde{\mathcal{U}}$$

$$\times \underbrace{\begin{bmatrix} 1 & & & & \\ & \frac{c_1}{z_1} & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \frac{c_d}{z_d} \end{bmatrix}}_{E}$$

Let $\tilde\eta(z) = \begin{bmatrix} 0 \\ z \\ 2z^2 \\ \vdots \\ Nz^N \end{bmatrix}$

$$J_P^{+} = E^{-1}\,\tilde{\mathcal{U}}^{+}$$

$\Rightarrow$ so we have reduced the problem to row sums of $\tilde{\mathcal{U}}^{+}$.

"Phase transition" $\quad \Delta = \min_{i \neq j} |z_i - z_j|$ (minimal separation)

well-conditioned regime. $\quad N > \frac{\text{const}}{\Delta}$

$$\tilde{\mathcal{U}}^{+} = (\tilde{\mathcal{U}}^{*}\tilde{\mathcal{U}})^{-1}\,\mathcal{U}^{*}$$

Let's investigate $(\tilde{\mathcal{U}}^{*}\tilde{\mathcal{U}})$.

We have several types of entries

(i) $\eta^*(z_i)\eta(z_i) = N+1 \quad \left(= \sum_{k=0}^{N} 1\right)$

(ii) $\eta^*(z_i)\tilde{\eta}(z_i) = \sum_{k=0}^{N} k = \frac{N(N+1)}{2}$

(iii) $\tilde{\eta}^*(z_i)\tilde{\eta}(z_i) = \sum_{k=0}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$

(iv)-(vi) $\sum_{k=0}^{N} k^q z^k$, $q = 0,1,2$, $z = z_j^* z_i = e^{it}$

$(0^0 = 1)$ $(t \neq 0)$

Summation by parts: $\{A_k\}$, $\{B_k\}$

$$\sum_{k=0}^{N} A_k(B_{k+1} - B_k) = A_{N+1}B_{N+1} - A_0 B_0 - \sum_{k=0}^{N} B_{k+1}(A_{k+1} - A_k)$$

$$\left(\int u\,dv = uv - \int v\,du\right)$$

$\{z = e^{it}\}$

$$A_k = k^q, \quad B_k = 1 + z + \dots + z^{k-1} \Rightarrow B_{k+1} - B_k = z^k, \; B_0 = 0$$

$$\Rightarrow \sum_{k=0}^{N} k^q z^k = (N+1)^q B_{N+1} - \sum_{k=0}^{N} \left\{(k+1)^q - k^q\right\} B_{k+1}$$

Check: $|B_k|^2 = \left(\frac{\sin\frac{(k+1)}{2}t}{\sin t/2}\right)^2 \leq \left(\frac{2}{t}\right)^2$

$$|B_k| \leq \frac{2}{|t|}$$

$$\Rightarrow \left| \sum_{k=0}^{N} k^q z^k \right| \le \frac{2}{|t|} \left\{ (N+1)^q + \sum_{k=0}^{N} \left\{ (k+1)^q - k^q \right\} \right\}$$

$$\le \frac{4}{|t|} (N+1)^q .$$

$$\tilde{u}^* \tilde{u} = \begin{array}{cc} & 2d \\ 2d & \begin{bmatrix} \boxed{B_0} & & B_{ij} \\ & \boxed{B_0} & \\ & & \ddots \end{bmatrix} \end{array} \qquad B_0 = \begin{bmatrix} N+1 & \frac{(N+1)N}{2} \\ \frac{(N+1)N}{2} & \frac{N(N+1)(2N+1)}{6} \end{bmatrix}$$

$$B_{ij} = \frac{1}{\Delta} O \begin{bmatrix} O(1) & O(N) \\ O(N) & O(N^2) \end{bmatrix}$$

big O notation

$$B_0^{-1} = \begin{bmatrix} \frac{2(N+1)}{N(N-1)} & \frac{-6}{N^2-N} \\ -\frac{6}{N^2-N} & \frac{12}{N^3-N} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} B_0 & & O \\ & B_0 & \\ & \ddots & \\ O & & B_0 \end{bmatrix}}_{S} \begin{bmatrix} I & & B_0^{-1} B_{ij} \\ & I & \\ & \ddots & \\ & & I \end{bmatrix} = S \Big(I_{2d \times 2d} - \underbrace{\begin{bmatrix} 0 & & B_0^{-1} B_{ij} \\ & \ddots & \\ & & 0 \end{bmatrix}}_{Y} \Big)$$

$$= S(I - Y)$$

$$(\tilde{u}^* \tilde{u})^{-1} = (I - Y)^{-1} S^{-1}$$

when is $(I-Y)$ invertible?

„like" $\frac{1}{1-x} = 1 + x + x^2 + \dots$ $(|x| < 1)$

Neumann series (of operators):

If $\sum_{k=1}^{\infty} Y^k$ converges (in the operator norm),

then $(I - Y)^{-1} = I + \sum_{k=1}^{\infty} Y^k$

$$C_{ij} = B_0^{-1} B_{ij} = \frac{1}{\Delta} \mathcal{O}\left(\begin{bmatrix} \frac{1}{N} & 1 \\ \frac{1}{N^2} & \frac{1}{N} \end{bmatrix}\right)$$

$$C_{ij} C_{kl} = \frac{d}{\Delta^2} \mathcal{O}\left(\begin{bmatrix} N^{-2} & N^{-1} \\ N^{-3} & N^{-2} \end{bmatrix}\right)$$

$$Y^k \propto \left[\quad C_{ij}^{(k)} \quad \right] \qquad C_{ij}^{(k)} = \frac{1}{d} \frac{d^k}{\Delta^k} \mathcal{O}\begin{pmatrix} N^{-k} & N^{-(k-1)} \\ N^{-(k+1)} & N^{-k} \end{pmatrix}$$

$$= \frac{1}{d} \left(\frac{d}{N\Delta}\right)^k \mathcal{O}\begin{pmatrix} 1 & N \\ N^{-1} & 1 \end{pmatrix}$$

$\Rightarrow \sum_{k=1}^{\infty} Y^k$ converges when $\frac{d}{N\Delta} \leq \text{const} < 1$

$$\Rightarrow \sum_{k=1}^{\infty} Y^k = \begin{bmatrix} Q & & R_{ij} \\ & \ddots & \\ & & Q \end{bmatrix} \times \frac{1}{1 - \frac{d}{N\Delta}},$$

$$Q = \mathcal{O}\begin{pmatrix} N^{-2} & N^{-1} \\ N^{-3} & N^{-2} \end{pmatrix}, \quad R_{ij} = \mathcal{O}\begin{pmatrix} N^{-1} & 1 \\ N^{-2} & N^{-1} \end{pmatrix}$$

$$\left(\tilde{\mathcal{U}}^{*}\tilde{\mathcal{U}}\right)^{-1} = \mathcal{O}\begin{pmatrix} \begin{bmatrix} N^{-1} & N^{-2} \\ N^{-2} & N^{-3} \end{bmatrix} & & L.O.T \\ & \ddots & \\ & & \ddots \end{pmatrix}$$

$$\tilde{\mathcal{U}}^{+} = \mathcal{O}\begin{array}{l} \partial/\partial c_j \rightarrow \\ \\ \end{array}\underbrace{\begin{pmatrix} N^{-1} & \cdots & - \\ N^{-2} & \cdots & - \\ & & \end{pmatrix}}_{n+1}\begin{array}{l} \\ \leftarrow \partial/\partial z_j \\ \end{array}$$

Final estimate:

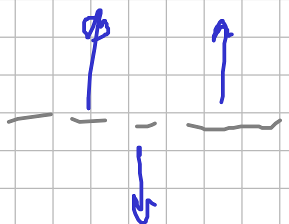
$\rightsquigarrow$ when $N > \frac{\text{const}}{\Delta}$, then

(well-conditioned regime) $\ell_j(x) = \begin{cases} \mathcal{O}(1) & \text{for } |c_j - \hat{c}_j| \\ \mathcal{O}\left(\frac{1}{N}\right) & \text{for } |z_j - \hat{z}_j| \end{cases}$

NOT REALLY „super-resolution"

For super-resolution: „super-localization"

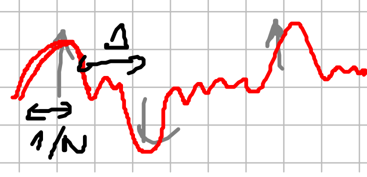
$$f = \sum c_j \delta(t - t_j) \longrightarrow c_k(f), \; k = 0, \ldots, N$$



$\longrightarrow (D_N * f)(t)$

$1/N \leq \Delta$

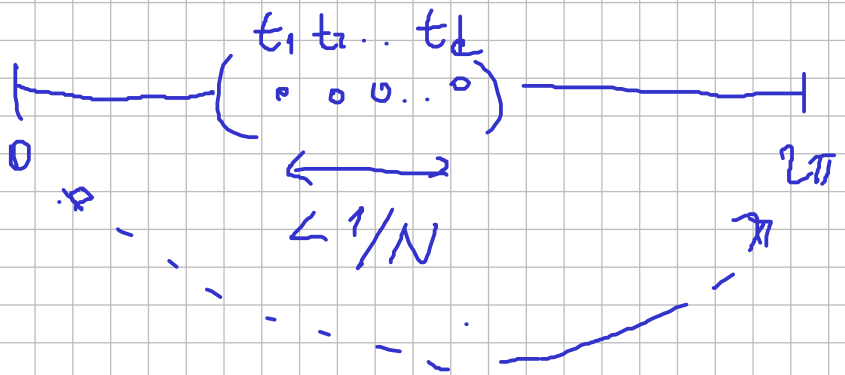
width of the D_N is $\approx \frac{1}{N}$



Super-resolution regime

$$\Delta \ll 1/N$$

<u>Case 1</u>: „Single cluster" $\left(z_j = e^{it_j}\right)$



$$\min_{i\neq j} |z_i - z_j| = \Delta$$

$$\max_{i\neq j} |z_i - z_j| < \frac{c_1}{N}.$$

$$\{c_j, t_j\} \to \left\{\sum_{j=1}^{d} c_j e^{i\omega t_j} + e(\omega)\right\}_{\omega\in[0,\Omega]}, \quad \|e\|_\infty \le \varepsilon$$

Idea: choose λ ("decimation" parameter, "blowup")

$$\lambda \in \left(0, \frac{\Omega}{2d-1}\right]$$

and consider

$$P_\lambda : \{c_j, z_j\} \to \left\{\sum_{j=1}^{d} c_j z_j^{\lambda k}\right\}_{k=0,\ldots,2d-1} = \left\{\sum_{j=1}^{d} c_j w_j^{k}\right\}_{k=0,\ldots,2d-1}$$

$$w_j = e^{i\lambda t_j}$$

In a single cluster case:

$$\min |w_j - w_i| = \lambda\Delta$$

$$\max |w_i - w_j| < \pi$$

$\Rightarrow$ We can apply local stability from lecture 9:

$\Rightarrow$ For $\varepsilon \ll 1$ small enough:

$$|\tilde{c}_i - c_i| \lesssim \left(\frac{1}{\lambda\Delta}\right)^{2d-1} \varepsilon$$

$$|\omega_i - \tilde{\omega}_i| \lesssim \frac{1}{|c_i|} \left(\frac{1}{\lambda\Delta}\right)^{2d-2} \cdot \varepsilon$$

$$z_j^{\lambda} = \omega_j \quad \Longrightarrow \quad z_j = (\omega_j)^{1/\lambda} \qquad (*)$$

$$\Rightarrow |z_j - \hat{z}_j| \lesssim \frac{1}{|c_j|} \frac{1}{\lambda} \left(\frac{1}{\lambda\Delta}\right)^{2d-2} \cdot \varepsilon$$

We want to maximize λ

Can take $\lambda = \lambda_{max} = \frac{\Omega}{2d-1}$

$\Longrightarrow$

Can show these are optimal,

$$\begin{cases} |c_i - \hat{c}_i| \lesssim \left(\frac{1}{\Omega\Delta}\right)^{2d-1} \cdot \varepsilon \\ \\ \begin{array}{l} |z_i - \hat{z}_i| \\ |t_i - \tilde{t}_i| \end{array} \lesssim \overbrace{\frac{1}{|c_i|} \cdot \frac{1}{\Omega} \left(\frac{1}{\Omega\Delta}\right)^{2d-2}}^{E_i} \cdot \varepsilon \end{cases}$$

$\frac{1}{\Omega\Delta}$ = "super-resolution factor" $\quad \frac{1/\Omega}{\Delta} = \frac{\text{Nyqist}}{\Delta} \gg 1.$

(*) Possible issue:

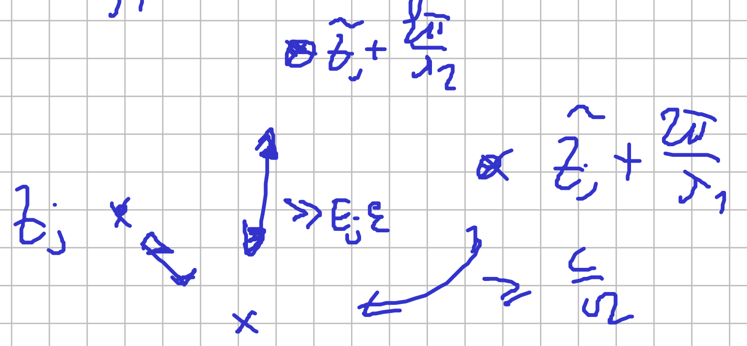
$$\omega_j = e^{i\lambda t_j}$$

$$(\lambda t_j) \equiv (\lambda t_j) + 2\pi \cdot n \qquad n \in \mathbb{Z}.$$

$$t_j \equiv t_j + \frac{2\pi}{\lambda} \cdot n \qquad \text{"spurious solutions".}$$

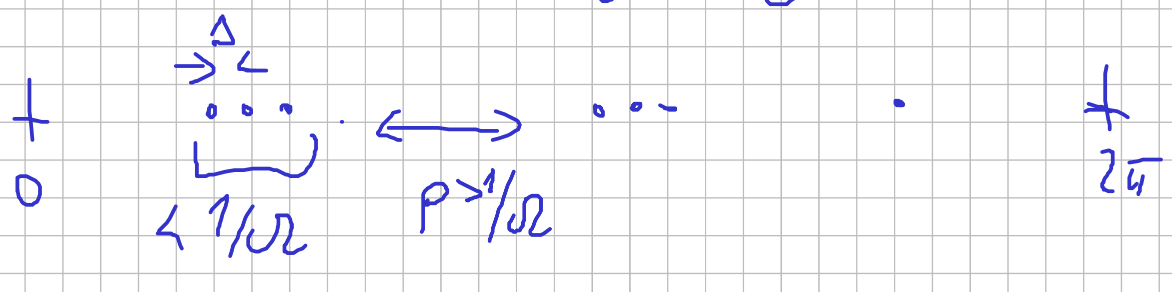
Apparently, no uniqueness!

$\lambda_1, \lambda_2 \approx O(\Omega)$

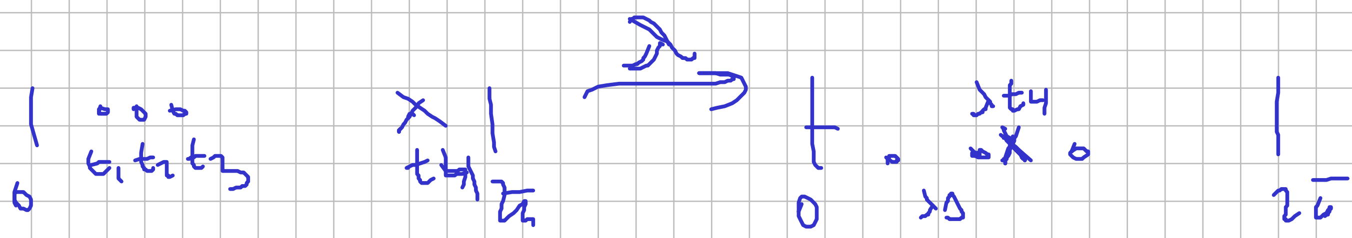


$$\tilde{z}_j = z_j + O(E_j \varepsilon)$$

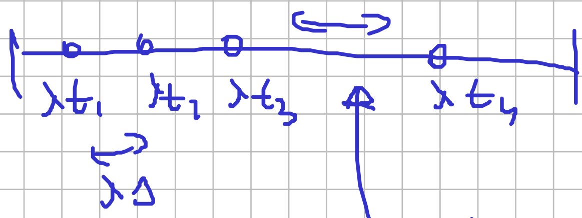
Multi-cluster geometry "Prior information".



We might have collisions



We want to have $\boxed{\lambda \approx O(\Omega)}$ (XX)



inter-cluster distance $\geq$ const

Then we would have

$$|c_i - \hat{c}_i| \leq \left(\frac{1}{\Omega\Delta}\right)^{2p_j - 1} \cdot \varepsilon$$

$$|t_i - \hat{t}_i| \leq \frac{1}{|c_i|}\frac{1}{\Omega}\left(\frac{1}{\Omega\Delta}\right)^{2p_j - 2} \cdot \varepsilon$$

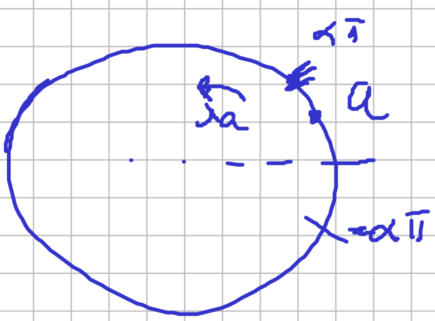
In particular, non-cluster nodes would become "perfectly stable"

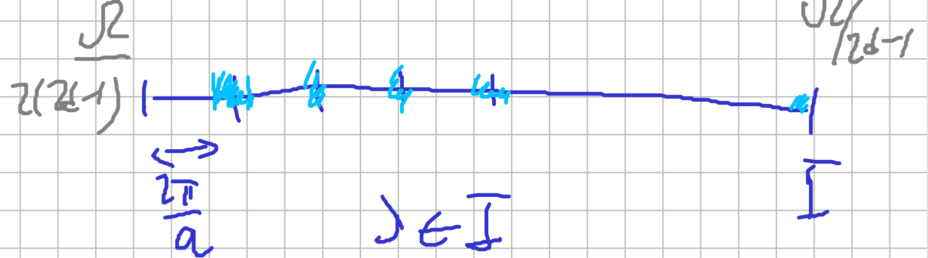
We will show that $\exists \lambda^*$ satisfying (XX)

Consider a pair of nodes t, y

$a = \|t-y\|_{\mathbb{T}}$ ← distance on the circle

$m\{\lambda : \|\lambda a\|_{\mathbb{T}} \leq \alpha \cdot \pi\} := \beta(\lambda)$





When λ traverses an interval of length $\frac{2\pi}{a}$, λa traverses the unit circle once. In particular, it falls into $[-\pi a, \pi a]$ exactly $\frac{2\alpha\pi}{2\pi} = \alpha$ "percent of time"

$$\beta(\lambda) \leq |I| \cdot \alpha.$$

Now put $I = \left[\frac{\Omega}{2(2d-1)}, \frac{\Omega}{2d-1}\right]$

$0 \leq \xi \leq 1$. We can ensure that all pairs remain separated by $\alpha = \frac{1-\xi}{d^2}$

Apply union bound:

$$m\{\lambda \in I : \text{at least one pair } (\epsilon, y) \text{ is } \alpha\text{-close after } \lambda\text{-blowup}\}$$

$$\leq \frac{\Omega}{2(2d-1)} \cdot \binom{d}{2} \cdot \alpha \leq |I| (1-\xi)$$

$$\Rightarrow m\{\lambda : \text{all pairs are separated}\} \geq \frac{\Omega}{2(2d-1)} \cdot \xi.$$

Not constructive: an algorith for choosing such λ is not known.