IP lecture 1 208 09/03/2020 ①

Hadamard's definition: a (direct/inverse) problem is well-posed if it
data ⇒ result
data ⇒ solution

1) has a solution;
2) the solution is unique;
3) the solution depends continuously on the data.

Classical examples by Hadamard: (1923)

consider the Cauchy problem for Laplace eqn

$\nabla^2 u(x,y) = 0$ $\begin{cases} u(x,0) = \frac{1}{n}\cos nx \\ \frac{\partial u}{\partial y}(x,0) = 0 \end{cases}$

→ unique solution $u(x,y) = \frac{1}{n}\cos nx \cosh ny$ $\left[\cosh x = \frac{e^x + e^{-x}}{2}\right]$

But when $n \to \infty$: $u(x,0) \to 0 \Rightarrow u(x,y) \to 0$ "should"

⇒ the solution is not continuous (in any reasonable topology).

⇒ so for many years people thought such examples to be "non-physical",

problems were called "incorrect", "improperly posed" etc.

[Turns out this problem appears in ECG :-)].

Another example: inverse heat propagation

Forward problem: $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $0 \le x \le 1$, $t \ge 0$; $u(x,0) = f(x)$ [initial heat distr.], $u(0,t) = u(1,t) = 0$

Solution by Fourier series expansion.

if $f(x) = \sum_{n=1}^{\infty} f_n \sin(\pi n x)$, $f_n = 2\int_0^1 f(x)\sin(\pi n x)\,dx$

(sep. of var.) $\rightsquigarrow u(x,t) = \sum_{n=1}^{\infty} f_n e^{-(\pi n)^2 t} \sin(\pi n x)$

② Smoothing / loss of information:

with finite precision ε (e.g. on a computer)
only N_ε coefficients f_n will be nonzero

But for $t = T$ (final time) this number might be much smaller ($\ll N_\varepsilon$) (due to the presence of $e^{-(\pi n)^2 t}$ factors).

Now consider the inverse: from $u(x,T)$ determine $u(x,0)$

Again, suppose that $u(x,T) = \frac{1}{n}\sin(\pi n x)$ then
$u(x,0) = \frac{1}{n}\sin(\pi n x)\, e^{(\pi n)^2 T}$

so when $n \to \infty$, $u(x,T) \to 0$ but $u(x,0) \to \infty$.

[Note: If we want $u(x,0) \to 0$ the initial $u(x,T)$ must be extremely smooth, $f_n \to 0$ super-exponentially]

this is not expected to occur in practice

But this shows that the property of continuity crucially depends on the spaces / norms chosen]

Integral equations

In the last example:

$u(x,0) = \sum f_n \sin(\pi n x)$

$u(x,t) = \sum f_n \sin \pi n x\, e^{-(n\pi)^2 t} = \int_0^1 u(y,0)\, k(y,x)\,dy$ where

$f_n = 2\int_0^1 u(y,0)\sin \pi n y\,dy$

$k_t(x,y) = 2\sum e^{-(\pi n)^2 t}\sin(\pi n x)\sin(\pi n y)$

(modern "hot" topics: manifold learning) → (heat kernel) (example of Green function)

This is a general form of problems of the form

$\boxed{\begin{array}{l} \text{direct: } \int_0^1 k(s,t) f(s)\,ds = g(t) \quad 0 \le t \le 1 \\ \text{inverse: find } f \text{ given } g \text{ (and } K) \end{array}}$ appear in many applications

③ More generally, we want to analyze operator equations

$Af = g$

$Ax = y$, $x \in X$, $y \in Y$
function spaces

So we need some functional analysis (but not too heavy...)

Luckily, when A is an integral operator
$Af = \int k(s,t) f(s)\,ds$

it behaves "like an infinite matrix" with diagonal elements → 0.

Hilbert space setting (rigorous treatment ⇒ FMA class)

X is a Hilbert space of functions

$X = \{ f : [0,1] \to \mathbb{R} \text{ (or } \mathbb{C}) : \int_0^1 |f|^2 dt < \infty \} = L^2[0,1]$.

Inner product: $f, g \in X$: $\langle f, g \rangle = \int f \bar{g}$

completeness: every Cauchy seq. converges to an elt. of the space

norm: $\|f\| = \sqrt{\langle f, f \rangle} \ge 0$
$\|f\| = 0$ iff $f = 0$ a.e.

Bounded operators X, Y Hilbert spaces (or Banach sp.)

$A : X \to Y$ bounded (bdd) if $\exists C > 0$ s.t. $\|Ax\|_Y \le C\|x\|_X$ $\forall x \in X$.

operator norm: $\|A\| := \sup_{\|x\| \le 1} \|Ax\| = \sup_{x \ne 0} \frac{\|Ax\|}{\|x\|}$

unbounded operators are much more difficult to treat [but e.g. Quantum Mechanics has a lot of them..]

Example: derivative op.

④ Continuous operators: if $x \to x_n$ (i.e. $\|x - x_n\| \to 0$ NOT pointwise) then $Ax_n \to Ax$.

Lemma 1 A linear op is continuous iff it is continuous at 0 (or any other point)

Lemma 2 A linear op. $A : X \to Y$ is bounded iff it is continuous.

Pf: ⇒: A bdd ⇒ $\|Ax_n\| \le C\|x_n\| \xrightarrow[\|x_n\| \to 0]{} 0$

⇐: Suppose A is unbounded, then $\exists \{x_n\}$ with $\|x_n\| = 1$ but $\|Ax_n\| \ge n$

Define $y_n = \frac{x_n}{\|Ax_n\|}$

⇒ $\|y_n\| \to 0$ but $\|Ay_n\| = 1$.

Compact operators: A is compact if $\forall \{x_n\}$ bdd, $\{Ax_n\}$ contains a converging subsequence.

This is an extremely important notion. Effectively, analysis of compact operators can be reduced to linear algebra.

Non-compact example: identity op. (provided X is infinite dimensional)

Orthonormal bases: we assume there exists an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ of X (then the space is called separable)
$\langle \varphi_k, \varphi_\ell \rangle = \delta_{k\ell} = \begin{cases} 1 & k = \ell \\ 0 & k \ne \ell \end{cases}$

[examples on $L^2(0,1)$: Fourier bases, orthogonal polynomials, ..].

So let $\{\varphi_n\}$ be the bounded sequence, then clearly $\{I\varphi_n\}$ does not contain a converging subsequence
$\left[\|\varphi_n - \varphi_m\| = \langle \varphi_n - \varphi_m, \varphi_n - \varphi_m \rangle = \|\varphi_n\|^2 + \|\varphi_m\|^2 = 2 \right]$.

[Remark: compact ⇒ bounded]
(but not vice versa!)

⑤ Adjoint operators $A : X \to Y$, X, Y Hilbert spaces

A bdd ⇒ ∃ unique $A^* : Y \to X$ s.t.

$\forall x \in X, y \in Y$: $\langle Ax, y \rangle_Y = \langle x, A^* y \rangle_X$

Lemma 3 let $k(s,t) \in L^2([0,1] \times [0,1])$, k (continuous)
(i.e. $\int_0^1\int_0^1 k^2(s,t)\,ds\,dt < \infty$)

Then the op. $A : L^2[a,b] \to L^2[c,d]$ given by

$Af = \int_a^b k(s,t) f(s)\,ds$, $c \le t \le d$

is a (bounded) compact operator.

Its adjoint is given by

$A^* g = \int_c^d k(s,t) g(t)\,dt$ $a \le s \le b$

Definition: a bdd op A is of finite rank if its range (i.e. the linear subspace $\{Ax : x \in X\}$) is finite-dimensional.

[Example: projection onto a finite # coordinates].

Proposition: If $\{A_n\}$ are compact and $A_n \xrightarrow[n \to \infty]{} A$ (in the operator norm) then A is compact.

Proof of lemma 3: 2) consider the sequence of finite rank operators $K_n(s,t) = \sum_{k=1}^{n}\sum_{\ell=1}^{n} c_{k\ell}\,\varphi_k(x)\psi_\ell(y)$

$A_n f = \int K_n(s,t) f(s)\,ds$

1) let $K(s,t) = \sum c_{k\ell}\,\varphi_k(x)\psi_\ell(y)$
$\sum_{k,\ell} |c_{k\ell}|^2 < \infty$
where $\{\varphi_k\}$ is an orth. basis for $L^2(a,b)$
$\{\psi_\ell\}$ — " — " — " $L^2(c,d)$

⑥ Proposition: $\left.\begin{array}{l} A : X \to Y \text{ compact} \\ B : Y \to Z \text{ bounded} \end{array}\right\} \Rightarrow BA$ is compact ($: X \to Z$)

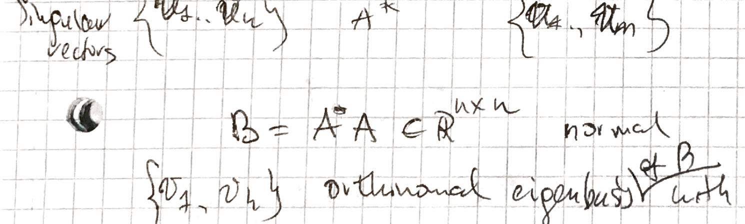
Corollary: If $A : X \to Y$ is compact then A^{-1}, if exists, cannot be continuous (bounded).

Important tool - Operator SVD

Reminder: A normal ($AA^* = A^*A$) $\in \mathbb{R}^{n \times n}$ iff A has a complete set of orthonormal eigenvectors $\{v_j\}$
$Av_j = \lambda_j v_j$

Singular value decomposition:

$A \in \mathbb{R}^{m \times n}$ ($m \ge n$) $A : \mathbb{R}^n \to \mathbb{R}^m$



$B = A^*A \in \mathbb{R}^{n \times n}$ normal

$\{v_1, .. v_n\}$ orthonormal eigenbasis of B with e.values $\{\lambda_i\}$

Take $\lambda_j \ne 0$, Define $u_j := \frac{Av_j}{\|Av_j\|}$

⇒ $\forall x \in \mathbb{R}^n$: $x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i + Qx$, $Q : \mathbb{R}^n \to N(A)$

$Ax = \sum_{i=1}^{n} \langle x, v_i \rangle u_i \|Av_i\|$

$Bv_i = \lambda_i v_i = A^*Av_i = A^* u_i \|Av_i\|$

$AA^* u_i \|Av_i\| = \lambda_i Av_i = \lambda_i \|Av_i\| u_i$

$A^* u_i = \frac{1}{\|Av_i\|} A^*Av_i = \frac{\lambda_i}{\|Av_i\|} v_i$

$\langle Av_i, u_i \rangle = \|Av_i\|$
$= \langle v_i, A^* u_i \rangle = \langle v_i, \frac{\lambda_i}{\|Av_i\|} v_i \rangle$
$= \frac{\lambda_i}{\|Av_i\|} \Rightarrow \|Av_i\| = \sqrt{\lambda_i}$

⑦ SVD $\mu_i = \sqrt{\lambda_i}$

$\boxed{Ax = \sum_{i=1}^{n} \mu_i \langle x, v_i \rangle u_i}$

$A = U \begin{bmatrix} \mu_1 & & & \\ & \ddots & & \\ & & \mu_n & \\ & & & 0 \end{bmatrix} V^*$

orth. column orth. columns

Analogue of eigendecomp.

Spectral theorem for compact, self-adjoint operators

$A : X \to X$, $A^* = A$ (X - separable Hilbert sp.)

Then ∃ orth. basis for X, $\{\varphi_k\}$, and a sequence $\lambda_k \to 0$

s.t. $A\varphi_k = \lambda_k \varphi_k$, $k = 1, 2, ..$

All eigenspaces for $\lambda \ne 0$ are finite-dimensional.

$\{\lambda_k\}$ is called the spectrum
($0 \in Sp(A)$ if A has a non-trivial kernel).

k plays the role of "frequency"

Proof: FMA class

Theorem (operator SVD)

Let $A : X \to Y$ compact. Let $\{\mu_i\}$ be the nonzero singular values (square roots of the nonzero eigenvalues of A^*A), repeated according to multiplicity:

$\text{mult}(\mu_i) = \dim N(\mu_i^2 I - A^*A)$.

Then ∃ orth. sequences $\{\varphi_n\} \subset X$, $\{\psi_n\} \subset Y$ s.t.

right singular functions; left singular functions

$A\varphi_n = \mu_n \psi_n$, $A^* \psi_n = \mu_n \varphi_n$.

⑧ For each $\varphi \in X$ there holds the SVE (expansion)

$\varphi = \sum \langle \varphi, \varphi_n \rangle \varphi_n + Q\varphi$, where

$Q : X \to N(A)$ is the orthogonal projection onto $N(A)$,

and

$A\varphi = \sum \mu_n \langle \varphi, \varphi_n \rangle \psi_n$.

Proof: proceeding as in the finite-dim. case, take $\{\varphi_n\}$ to be the orthobasis of eigenfunctions (with nonzero eigenvalue μ_n^2) of A^*A, and put

$\psi_n = \frac{1}{\mu_n} A\varphi_n$.