Lecture 6

I) Stable Analytic continuation

setup:

f, analytic in $D = \{ z \in \mathbb{C} : |z| < 1 \}$

$\Gamma \subset D$ smooth arc

Given: f on Γ (data can be noisy)

Find: f in D.

From standard complex analysis: we know there is a unique solution (if $f \equiv 0$ on Γ then $f \equiv 0$)

In the noiseless case: classical solution by re-expansion of the Taylor series coefficients.

Weierstrass method

Ref: Henrici vol I, Th 3.6

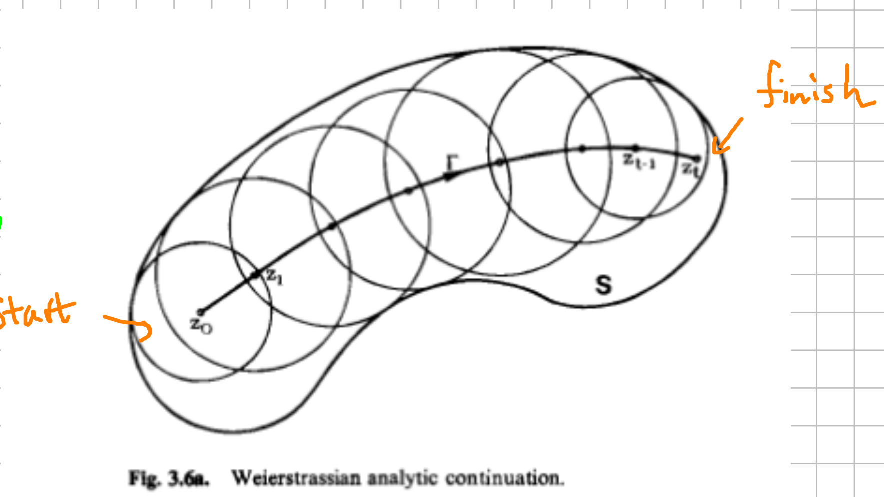


Fig. 3.6a. Weierstrassian analytic continuation.

$$\sum_{n\geq 0}^{N_1} a_n(z_0)(z-z_0)^n =$$

$$\sum_{n\geq 0}^{N_2} a_n(z_1)(z-z_1)^n =$$

$$\cdots$$

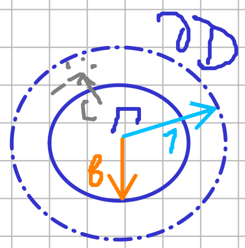
$$\sum_{n\geq 0}^{N_t} a_n(z_t)(z-z_t)^n$$

For fixed t, can choose $N_1 \geq \ldots \geq N_t \to \infty$, Process will converge.

What if measurements are noisy?

Problem is ill-posed!

Example:



$$\Gamma = \{ |z| = b,\ 0 < b < 1 \}$$

$$f_n = \left(\frac{z}{c}\right)^n, \quad b < c < 1$$

$$\|f_n\|_\Gamma \to 0 \text{ as } n \to \infty, \text{ but } \|f_n\|_{\partial D} \to \infty$$

A-priori bound: $\|Bf\| = \|f\|_{\partial D} \le E$.

Consider the inverse problem

forward operator

$$A : \underbrace{A_C(D)}_{\text{analytic f-ns in } D} \to L_\infty(\Gamma), \qquad f(z) = \sum_{n\ge 0} c_n z^n$$

(representation in a basis)

$$Af = f|_\Gamma$$

constraint ~~restriction~~ operator

$$Bf(\theta) = f(e^{i\theta}) = \sum_{n \ge 0} c_n e^{in\theta}$$

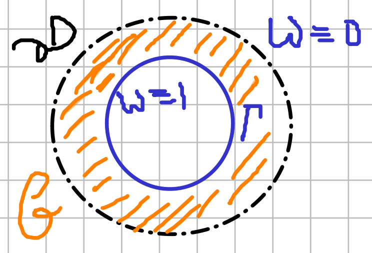
Recall from Lecture 2: worst-case error

$$M_z = \sup\{ |f(z)| : \|Af\| \le \varepsilon,\ \|Bf\| \le E \}$$

not a norm but a semi-norm, still OK

Theorem: Let $\omega(z)$ be the solution to the Dirichlet problem in the domain G with boundary values:

Here D, Γ arbitrary!



$$\omega\big|_{\partial D} = 0$$

$$\omega\big|_{\Gamma} = 1$$

Then

$$M_z \le \left(\frac{\varepsilon}{E}\right)^{\omega(z)} \cdot E, \quad z \in D$$

Proof: Let f satisfy $\|Af\| \le \varepsilon$, $\|Bf\| \le E$.
f analytic in D

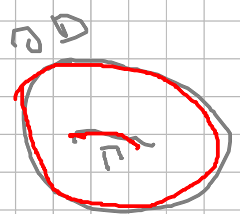
- $\log|f|$ is subharmonic (always)
  $= \mathrm{re}(\log f)$

- On Γ: $\log|f| \le \overset{=1}{\omega(z)} \log \varepsilon$ $\qquad f \le \varepsilon$
- on ∂D: $\log|f| \le (1-\underset{=0}{\omega(z)}) \log E$ $\qquad f \le E$

$$\Rightarrow \log|f| \le \omega(z)\log\varepsilon + (1-\omega(z))\log E \quad := u(z)$$

on $\cancel{\partial D} \setminus \Gamma$ $\qquad$ $\uparrow$ $u(z)$ is harmonic in D

∂G



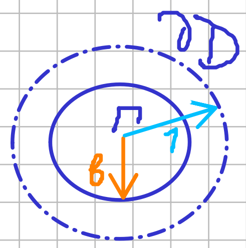
$\Longrightarrow$ by the maximum principle, the inequality holds in D.

For „nice" domains, can have formula for $\omega(z)$.

Back to our example ...

Textbook on potential theory:

$$\omega(z) = \frac{\log|z|}{\log 6}$$



Consider $f_N = E \cdot z^N$ with N large enough

so that $E \cdot 6^N < \varepsilon \Rightarrow N = \frac{\log \frac{\varepsilon}{E}}{\log 6}$

Then $f_N(z) = E$ on ∂D, $\Rightarrow f_N = E \cdot z^{\frac{\log \frac{\varepsilon}{E}}{\log 6}}$

$$\Rightarrow \log|f_N| = \log E + \frac{\log \varepsilon/E}{\log 6} \log|z| = \log E + \frac{\log|z|}{\log 6} \log \frac{\varepsilon}{E}$$

For $z \in D$, $|f_N(z)| = E\left(\frac{\varepsilon}{E}\right)^{\omega(z)}$

$\Rightarrow f_{N(\varepsilon,E)}$ attains the worst-case bound.

This example is very typical.

The general scheme

- Represent $f(t) \sim \sum_{n \geq 0} c_n \varphi_n(t)$ on Γ (*)

appropriately chosen, e.g. orthogonal poly's

Then (*) automatically defines an analytic extension.

- Find optimal truncation $N = N(\varepsilon)$ depending on the problem.

- Find approximate coefficients

$$\{\tilde{c}_n\}_{n \leq N(\varepsilon)}$$

based on the data. $\{f(t_i) + \varepsilon_i\}_{t_i \in \Gamma}$.

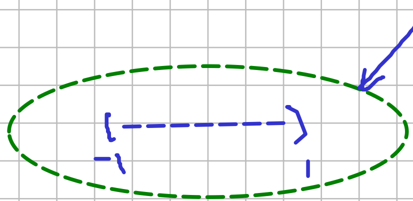
- Additional issue: estimate optimal # samples. $\{t_i\}_{i=1}^{M(\varepsilon, ..)}$

Additional examples

- $\Gamma = [-1,1]$ [Demanet & Townsend]

  $\varphi_n = T_n$ (Chebyshev)

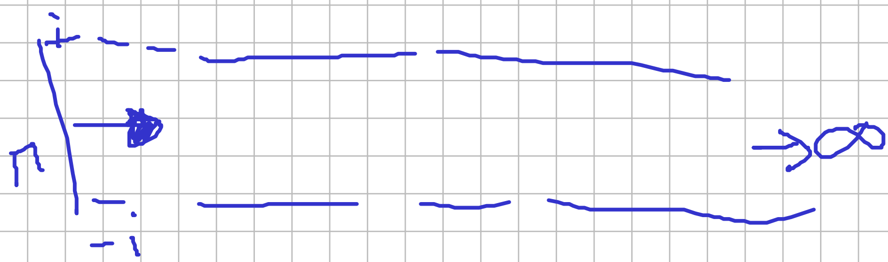
  $\Rightarrow$ extend to B_R (Bernstein ellipse)



Result: $r(x) = \dfrac{x+\sqrt{x^2-1}}{R}$, $\alpha(x) = \dfrac{-\log r(x)}{\log R}$

$$\Rightarrow \quad |\tilde{f}(x) - f(x)| \leq C_{R,\varepsilon} \frac{E}{1-r(x)} \left(\frac{\varepsilon}{\bar{\varepsilon}}\right)^{\alpha(x)}$$

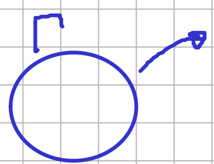
- Half-strip [Trefethen 2020]



$$|\tilde{f}(x) - f(x)| \leq \varepsilon^{C_1 e^{-C_2 x}}, \quad x > 0$$

- Upper half-plane [Grabovsky & Hovsepyan 2020]

$\mathrm{Im}(z) > 0$



II) Super-resolution and AC.

- We can try and extrapolate $\hat{f}(\omega)$ directly
- Need to know analyticity properties of FT.

Paley-Wiener theorem(s).

Provides the link between analyticity of $\hat{f}(\omega)$ and support/decay of f.

Idea: extend the Fourier formula to $\mathbb{C}$

$$F(\omega) = \int_{\mathbb{R}} f(t) e^{it\omega}\, dt \quad , \omega \in \mathbb{R}$$

$$\Rightarrow F(z) = \int_{\mathbb{R}} f(t) e^{izt}\, dt \quad , z \in \mathbb{C}$$

May have a problem with convergence if e.g. $\operatorname{Im} z < 0$ and $t < 0$.

- If $\operatorname{supp} f \subset \mathbb{R}^+$ then $F(z)$ is holomorphic in the upper half-plane.
- If $\operatorname{supp} f \subset [-T, T]$ then $F(z)$ is entire of exponential type T, i.e.

$|g(z)| \le e^{\alpha |z|^\rho}$

$$|f(z)| \le C\, e^{T|z|} \quad , z \in \mathbb{C}.$$

PW: converse is true as well.

"Ideal" SR via extrapolation

Given: $\hat{f} \in B_T$ $(=PW_T)$, a F.T. of a compactly supported f-n. on $|\omega| \le \Omega$

Extend $\hat{f}$ to $\mathbb{R}$.

Solution: use the PSWF basis. (with parameter $c=T$).

What about other filters, e.g.

$$\hat{f}_{\text{filtered}} = \hat{f} \cdot \hat{K} \quad \text{where}$$

$\text{supp}\, \hat{K} \subset [-1,1]$

or just $\hat{K}(\omega)$ rapidly decaying?

„Soft extrapolation"

Setup: $\sigma > 0$, $g(\omega) = \int_{-\sigma}^{\sigma} f(t) e^{i\omega t} dt$

$\Rightarrow |g(z)| \sim e^{\sigma |z|}$

Equisparced grid:

$$\tilde{h}(\omega_k) = g(\omega_k) \underbrace{W(\omega_k)}_{e^{-\omega^2/2}} + \underbrace{e(\omega_k)}_{O(\varepsilon)}, \quad |e(\omega_k)| < \varepsilon.$$

$W(\omega) = e^{-\omega^2/2}$ $\qquad \omega_k = k \Delta\omega, \quad |k| = 0, 1, .., N$

$\underline{\omega} = \{\omega_k\}$ $\qquad \omega_k \in [-N\Delta\omega, N\Delta\omega]$

Solve:

$$P_M = \operatorname*{argmin}_{p \in P_M} \| \hat{h}(\underline{\omega}) - p(\underline{\omega}) \|_W^2$$

$$\|\underline{x}\|_W^2 = \underline{x}^T [\operatorname{diag} W(\omega_k)] \underline{x}$$

Question: how to choose $M, N, \Delta\omega$

s.t. $|g(z) - P_M(z)|$ is asympt. optimal?

Some details

$$\boxed{\begin{array}{ll} He & e^{-x^2} \\ & e^{-x^2/2} \end{array}}$$

Hermite orth. polynomials $\{H_n(\omega)\}$

$$g(\omega) = \sum_{n\geq 0} c_n H_n(\omega), \quad c_n = \int_{\mathbb{R}} g(\omega) e^{-\omega^2} H_n(\omega)\, d\omega$$

$$= \int_{\mathbb{R}} h(\omega)\, \varphi_n(\omega)\, d\omega$$

$\varphi_n(\omega) = W(\omega) H_n(\omega)$ a Hermite function, orthonormal on $\mathbb{R}$

LS solution: $\tilde{g}(\omega) = \sum_{n\leq M} \tilde{c}_n H_n(\omega)$

$$\tilde{\underline{c}}_n = V_E^{+} \underline{y} \qquad V_E = [\varphi_n(\underline{\omega})]_{n\leq M} \quad (2N+1)\times(M+1)$$

$$\underline{y} = \{\tilde{h}(\omega_k)\}$$

Step 0: $|C_n| \sim \exp\left(-\frac{n}{2}\log\frac{2n}{e\sigma^2}\right)$ $\qquad g = e^{-\omega}$ $\int$

Step 1: Estimate

$$\|\tilde{C}_n - C_n\| \sim \frac{\sqrt{N}}{\sigma_{min}(V_E)}\left(\varepsilon + |C_M|\right)$$

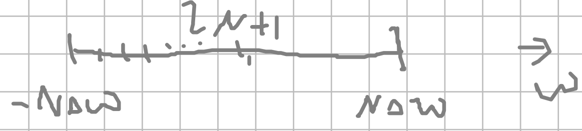
$\|y\| \sim \sqrt{N}\cdot\varepsilon$ $\qquad$ decay exp.

truncation to a finite interval

Step 2: when is $V_E^T V_E$ approximately diagonal?
$$\overset{\parallel}{\mathcal{U}}$$

when $\varphi_k(\underline{\omega})^T \varphi_\ell(\underline{\omega}) \approx \delta_{k\ell}$?

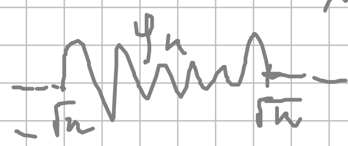
Balance between N, M



Some facts about Hermite pol. H_n

- Essential range: $[-\sqrt{n}, \sqrt{n}]$ $\quad \psi_n = H_n e^{-w^2/2}$
- Exp. decay

$N\Delta\omega \approx \sqrt{M}$



Idea: approximate $\int_{\mathbb{R}}$ by trapezoidal rule.

$$I = \int_{\mathbb{R}} \omega(x)\,dx, \quad I_h = h \sum_{k \in \mathbb{Z}} \omega(kh), \quad I_h^N = h \sum_{k=-N}^{N} \omega(kh)$$

Poisson Σ: $\quad I_h = 2\pi \sum_{j \in \mathbb{Z}} \hat{\omega}\left(\frac{2\pi}{h} j\right), \quad I = 2\pi \hat{\omega}(0)$

$\left\{ \begin{array}{l} \to \text{choose } h \text{ small enough s.t. } \hat{\omega}\left(\frac{2\pi}{h}\right) \text{ is small} \\ \to \text{choose } Nh \text{ large enough s.t. } \omega(Nh) \text{ is small} \end{array} \right.$

Result:

$$|I - I_h^N| \sim \mathcal{O}\left(e^{-\pi^2/h^2}\right) + \mathcal{O}\left(e^{-(Nh)^2}\right)$$

Optimal choice: $\underline{h \sim N^{-1/2}}$

$$M \sim N$$

Step 3: optimal M: $\quad C_{M_\varepsilon} \approx \varepsilon$

$$\Rightarrow M_\varepsilon = \frac{2 \log 1/\varepsilon}{W\left(\frac{4}{e\pi^2} \log \frac{1}{\varepsilon}\right)} \leftarrow \text{Lamberts } W \text{ fun.}$$

Step 4: Estimate

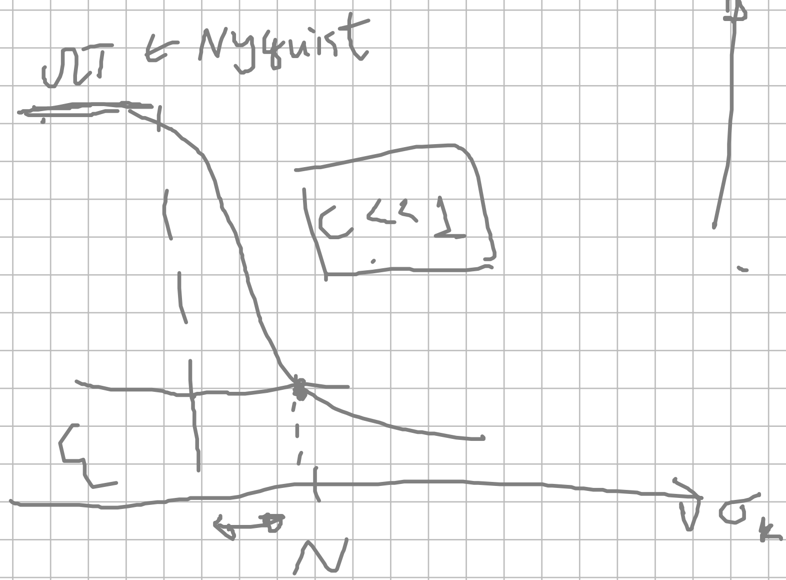
$$|g(z) - p_M(z)| \lesssim \underbrace{(\varepsilon + |C_M|)}_{\sim \varepsilon} \sum_{n \le M} |H_n(z)| + \underbrace{\sum_{n > M} |c_n H_n(z)|}_{T \ll 1}$$

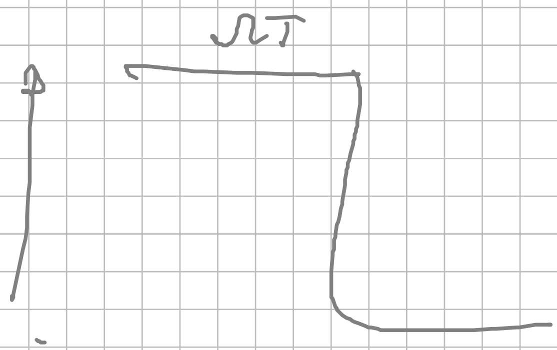
- Accurate estimates on growth of $H_n(z)$, $n \to \infty$, z fixed.

Szego's book:

$$|H_n(\omega)| \asymp \begin{cases} \exp\left(\frac{\omega^2}{2}\right) & n > \frac{\omega^2 - 1}{2}, \\ \exp\left\{\frac{\omega^2}{2}(1-\alpha)\left(1 + \frac{1+\alpha}{2}\log\frac{1+\alpha}{1-\alpha}\right)\right\} & n < \frac{\omega^2-1}{2}, \end{cases}$$

where $\alpha(n, \omega) = \sqrt{1 - \frac{2n+1}{\omega^2}}$. From these bounds it can be





$\sigma_N \approx \varepsilon$