Algebraic signal sampling, Gibbs phenomenon and Prony-type systems

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***Abstract*—Systems of Prony type appear in various signal reconstruction problems such as finite rate of innovation, super-resolution and Fourier inversion of piecewise smooth functions. We propose a novel approach for solving Prony-type systems, which requires sampling the signal at arithmetic progressions. By keeping the number of equations small and fixed, we demonstrate that such "decimation" can lead to practical improvements in the reconstruction accuracy. As an application, we provide a solution to the so-called Eckhoff's conjecture, which asked for reconstructing jump positions and magnitudes of a piecewise-smooth function from its Fourier coefficients with maximal possible asymptotic accuracy – thus eliminating the Gibbs phenomenon.**

I. Introduction

The "Prony system" of equations

$$m_k = \sum_{j=1}^{K} c_j z_j^k, \qquad c_j, z_j \in \mathbb{C},\ k \in \mathbb{N} \tag{1}$$

appeared originally in the work of R.Prony [18] in the context of fitting a sum of exponentials to observed data samples. He showed that the unknowns $\{c_j, z_j\}_{j=1}^K$ can be recovered explicitly from $\{m_0, \dots, m_{2K-1}\}$ by what is known today as "Prony's method". The system (1) appears in areas such as frequency estimation, Padé approximation, array processing, statistics, interpolation, quadrature, radar signal detection, error correction codes, and many more. In modern signal processing, (1) is of fundamental importance in the field of sub-Nyquist sampling (related terms are superresolution [9], [10] and finite rate of innovation [12]). A basic problem there is to recover an unknown "spike train", a linear combination of δ-functions

$$f(x) = \sum_{j=1}^{K} b_j \delta(x - x_j), \qquad c_j \in \mathbb{R},\ x_j \in [-\pi, \pi]$$

from its Fourier samples

$$\widehat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, \mathrm{e}^{-\imath k t}\, \mathrm{d}\, t. \tag{2}$$

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The resulting system is of course a special case of (1). If a more general model is considered,

$$f(x) = \sum_{j=1}^{K}\sum_{\ell=0}^{\ell_j-1} b_{\ell,j}\delta^{(\ell)}(x - x_j), \quad b_{\ell,j} \in \mathbb{R},\ x_j \in [-\pi, \pi], \tag{3}$$

then (2) becomes, after a change of variables,

$$m_k = \sum_{j=1}^{K} z_j^k \sum_{\ell=0}^{\ell_j-1} c_{\ell,j} k^{\ell}, \qquad c_{\ell,j} \in \mathbb{C},\ |z_j| = 1. \tag{4}$$

Many research efforts are devoted to stable solution of Prony-type systems (see e.g. [2], [8], [11], [17], [19] and references therein). We propose a novel approach to this problem, which requires sampling the signal at arithmetic progressions. By keeping the number of equations small and fixed, we demonstrate (in Section II) that such "decimation" can lead to practical improvements in the reconstruction accuracy, to a certain extent avoiding a well-known numerical instability of these systems.

In Section III we consider the problem of recovering a piecewise-smooth function, including the positions of its discontinuities, from its Fourier samples. The algebraic reconstruction method due to K.Eckhoff in essense required a solution of a particular instance of the system (4) with the error in the left-hand side having a certain asymptotic decay rate. Previously it was shown in [6], [7] that this approach yields a nonlinear approximation which is "half as accurate" compared to the best possible bound. As we elaborate in Section III, applying the decimation technique to the Prony-type system results in full asymptotic accuracy, thus completely eliminating the Gibbs phenomenon.

In Section IV we discuss several promising directions for future research.

II. Decimated Prony-type systems

Suppose that the "polynomial Prony model" (4) is to be fitted to the noisy measurements $\widetilde{m}_0, \dots \widetilde{m}_{M-1}$. We denote the number of unknowns by $R = \sum_{j=1}^{K}(\ell_j + 1)$. At first sight, using all the M measurements for fitting should improve reconstruction accuracy. While this is certainly justified in the case where the noise statistics are known (as demonstrated in e.g. [2], [19]), this might backfire if the noise is "adversary", or "worst-case". Potts & Tasche [17] show that when Prony system (1) is solved by least squares minimization for all M equations at once, then even if the nodes $\{z_j\}$ are detected very accurately, the error for magnitudes is amplified by a factor of $\sqrt{MR}$. This shows that it might actually be productive to stay with small number of measurements. We are therefore justified in making a simplifying assumption that the number of equations used for reconstruction equals the number of unknowns R. In this case the solution to the reconstruction problem can be characterized as the exact inversion of the measurement mapping $\mathcal{P}_I : \mathbb{C}^R \to \mathbb{C}^R$ which associates to any parameter vector $\mathbf{x} = \left\{\{c_{ij}\}, \{x_i\}\right\} \in \mathbb{C}^R$ its corresponding exact measurement vector $\mathbf{y} = \left(m_{i_0}, \dots, m_{i_{R-1}}\right) \in \mathbb{C}^R$ where $I = \{i_0 < i_1 < \dots < i_{R-1}\} \subset [0, M-1]$ is a given index set. Perhaps the most natural choice for the index sets I is given by arithmetic progressions

$$I_{t,p} = \{t, t+p, \dots, t+(R-1)p\}, \qquad t \geqslant 0,\ p \geqslant 1.$$

Following [8], we estimate for such $I = I_{t,p}$ the (local) stability of inversion by the *Lipschitz constant of* $\mathcal{P}_I^{-1}$ at the regular points of $\mathcal{P}_I$, which in turn are given by the following proposition.

Proposition 1. *The vector* $\mathbf{x} = (\{z_j, c_{i,j}\}) \in \mathbb{C}^R$ *is a regular point of* $\mathcal{P}_I$ *with* $I = I_{t,p}$ *if and only if* $z_j^p \neq z_i^p$ *for* $i \neq j$*, and* $c_{\ell_j-1,j} \neq 0$ *for all* $j = 1, \dots, K$.

We have the following upper bound on the accuracy of *any solution method*.

Theorem 2. *Consider the polynomial Prony system* (4) *with a fixed structure* $\left\{K, \{\ell_j\}_{j=1}^K\right\}$ *on* $I = I_{t,p}$*, and let* $\mathbf{x} = (\{z_j, c_{i,j}\}) \in \mathbb{C}^R$ *be a regular point of* $\mathcal{P}_I$*. If the error in each measurement is bounded in absolute value by* $\varepsilon \ll 1$*, then the errors in recovering the components of the original parameter vector* $\mathbf{x}$ *satisfy*

$$|\Delta c_{i,j}| \leq C(i,\ell_j)\left(\frac{2}{\delta_p}\right)^R \left(\frac{1}{2}+\frac{R}{\delta_p}\right)^{\ell_j} \frac{t^{\ell_j-i}}{p^i}\left(1+\frac{|c_{i-1,j}|}{\left|c_{\ell_j-1,j}\right|}\right)\varepsilon,$$

$$|\Delta z_j| \leq \frac{2}{\ell_j!}\left(\frac{2}{\delta_p}\right)^R \frac{1}{\left|c_{\ell_j-1,j}\right|} p^{-\ell_j}\varepsilon,$$

where $\delta_p \overset{def}{=} \min_{i\neq j}\left|z_j^p - z_i^p\right|$ *and* $C(i,\ell_j)$ *is an explicit constant (for consistency we take* $c_{-1,j} = 0$ *in the above formula).*

This result directly generalizes earlier stability estimates of [8] for the special case $I = I_{0,1}$. The proofs of both Proposition 1 and Theorem 2 are based on factorizing the Jacobian matrix of the map $\mathcal{P}_I$ along the same lines as in [8], while adding the analysis of the Jacobian's dependence on t and p.

Now suppose that the number of available measurements $M \to \infty$, while the noise ε remains bounded. It is easy to see that for the index set $I = I_{0,\lfloor \frac{M}{R} \rfloor}$ we obtain an improvement in accuracy of recovering the jump z_j of the order $\sim M^{\ell_j}$, compared with the non-decimated measurement set $I_{0,1}$.

Remark 3. If initially two nodes are close (say by δ), the decimation improves accuracy up to a certain limit. To see this, just substitute $\delta_p \sim p\delta$ into Theorem 2 and get an improvement by factor of $p^{-R-\ell_j}$.

Turning to particular solution methods, the decimation is fairly straightforward to implement. Indeed, taking any algorithm for the standard Prony-type system, one just needs to make the following modifications (for simplicity we consider only the recovery of the nodes $\{z_j\}$).

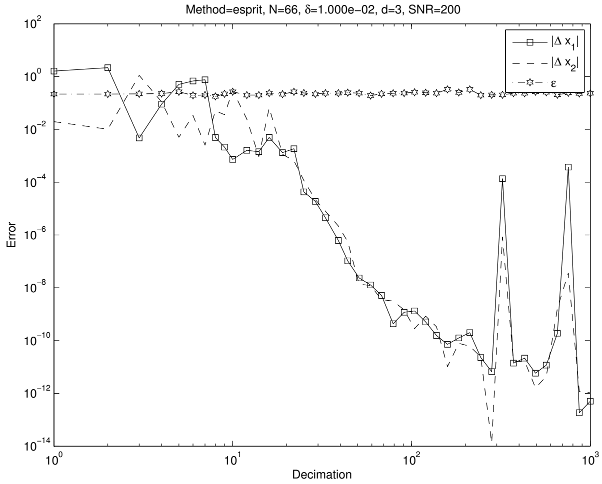
1) Choose the decimation parameter p.
2) Feed the original algorithm with the decimated measurements $m_0, m_p, m_{2p} \dots$, and obtain the estimated node w_j.
3) Take $z_j = \sqrt[p]{w_j}$.

We have tested the decimation technique according to the above procedure on two well-known algorithms for Prony systems - ESPRIT [2] and nonlinear least squares (LS, implemented by MATLAB's `lsqnonlin`). In the first experiment, we fixed the number of measurements to be 66, and changed the decimation parameter p, while keeping the noise level constant. The accuracy of recovery increased with p – see Figure 1 on page 3. In the second experiment, we fixed the highest available measurement to be $M = 2200$, and changed the decimation from $p = 1$ to $p = 100$ (thereby reducing the number of measurements from 2200 to just 22). The accuracy of recovery stayed relatively constant – see Figure 2 on page 3. Note that such a reduction leads to a corresponding decrease in the running time (calculating singular-value decomposition of large matrices, as in ESPRIT, is a time-consuming operation).

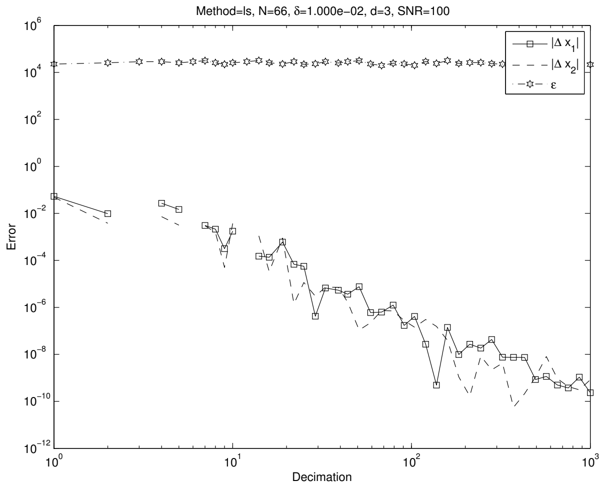
III. Piecewise-smooth Fourier reconstruction

Consider the problem of reconstructing an integrable function $f : [-\pi, \pi] \to \mathbb{R}$ from a finite number of its Fourier coefficients (2). If f is C^d and periodic, then the truncated Fourier series $\mathfrak{F}_M(f) \overset{\text{def}}{=} \sum_{|k|=0}^M c_k(f)\,\mathrm{e}^{\imath kx}$ approximates f with error at most $C \cdot M^{-d-1}$, which is optimal. If, however, f is not smooth even at a single point, the rate of accuracy drops to only M^{-1}. Still, one can hope to restore the best accuracy by using the a-priori information to produce some non-standard summation method. This accuracy problem, also known as the Gibbs phenomenon, is very important in applications, such as calculation of shock waves in PDEs. It has received much attention especially in the last few decades - see e.g. a recent book [16].

The so-called "algebraic approach" to this problem, first suggested by K.Eckhoff [13], is as follows. Assume that f has $K > 0$ jump discontinuities $\{x_j\}_{j=1}^K$, and $f \in C^d$ in every segment (x_{j-1}, x_j). We say that in this case f belongs to the class $PC(d, K)$. Denote the associated jump magnitudes at x_j by $a_{\ell,j} \overset{\text{def}}{=} f^{(\ell)}(x_j^+) - f^{(\ell)}(x_j^-)$. Then write the piecewise smooth f as the sum $f = \Psi + \Phi$, where $\Psi(x)$ is smooth and periodic and $\Phi(x)$ is a piecewise polynomial of degree d, uniquely determined by $\{x_j\}, \{a_{\ell,j}\}$ such that it "absorbs" all the discontinuities of f and its first d derivatives. In particular,

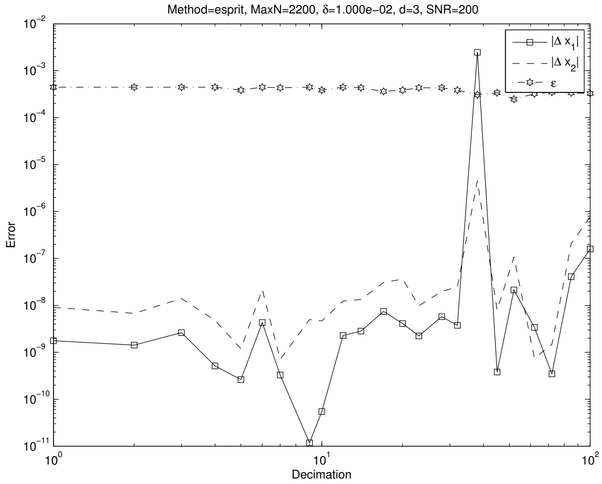


(a) ESPRIT, $d = 3$

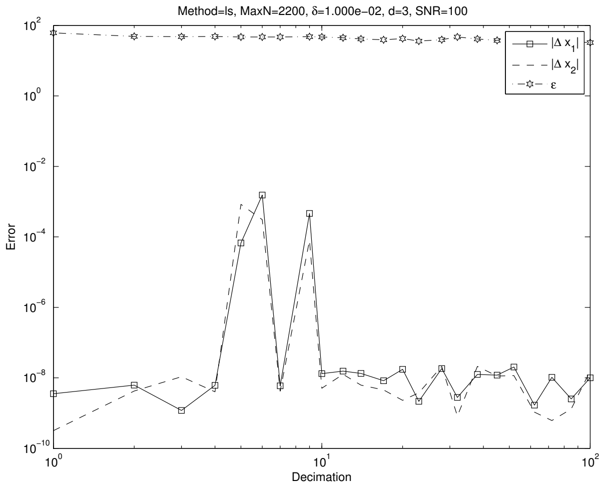


(b) LS, $d = 3$

Figure 1: Reconstruction error as a function of the decimation with fixed number of measurements ($M = 66$). The signal has two nodes with distance $\delta = 10^{-2}$ between each other. Notice that ESPRIT requires significantly higher Signal-to-Noise Ratio in order to achieve the same performance as LS.



(a) ESPRIT, $d = 3$



(b) LS, $d = 3$

Figure 2: Reconstruction error as a function of the decimation, reducing number of measurements from $M = 2200$ to $M = 22$. The signal has two nodes with distance $\delta = 10^{-2}$ between each other. The reconstruction accuracy remains almost constant.

the Fourier coefficients of Φ have the explicit form

$$c_k(\Phi) = \frac{1}{2\pi}\sum_{j=1}^{K} \mathrm{e}^{-\imath k x_j} \sum_{\ell=0}^{d} (\imath k)^{-\ell-1} a_{\ell,j}, \quad k = 1, 2, \ldots. \tag{5}$$

For $k \gg 1$, we have $|c_k(\Phi)| \sim k^{-1}$, while $|c_k(\Psi)| \sim k^{-d-2}$. Consequently, Eckhoff suggested to pick large enough k and solve the approximate system of equations (4) where $m_k = 2\pi(\imath k)^{d+1} c_k(f)$, $z_j = \mathrm{e}^{-\imath x_j}$ and $c_{\ell,j} = \imath^{\ell} a_{d-\ell,j}$. His proposed method of solution was to use the known values $\{m_k\}_{k \in I}$ where

$$I = \{M - (d+1)K + 1, M - (d+1)K + 2, \ldots, M\} \tag{6}$$

to construct an algebraic equation satisfied by the unknowns $\{z_1, \ldots, z_K\}$, and solve this equation numerically. Based on some explicit computations for $d = 1, 2;\ K = 1$ and large number of numerical experiments, he conjectured that his method would reconstruct the jump locations with accuracy M^{-d-1}.

Let us consider the problem in the framework of Prony type system (4). The error term is of magnitude $|\varepsilon| \sim M^{-1}$. The index set (6) is just $I_{t,p}$ with $t \sim M,\ p = 1$ (i.e. no decimation). Therefore, by Theorem 2 we get accuracy only of order $|\Delta x_j| \sim M^{-1}$.

Now consider the decimated setting for this problem. By the above, we can approximate eash jump x_j up to accuracy M^{-1}. Set

$$N = \left\lfloor \frac{M}{(d+2)K} \right\rfloor.$$

Now take the index set $I_{t,p}$ where $t = p = N$, i.e. $I_{N,N} =$

Algorithm 1 Full accuracy Fourier reconstruction of piecewise smooth functions

Let $f \in PC(d, K)$, and assume that $f = \Phi^{(d)} + \Psi$ where $\Phi^{(d)}$ is the piecewise polynomial absorbing all discontinuities of f, and $\Psi \in C^d$.

1) Obtain initial approximations for $\{x_1, \dots, x_K\}$ by any standard method (i.e. Eckhoff's method of order zero).
2) Localize each x_j by multiplying with a mollifier (convolution in Fourier domain).
3) Solve resulting Prony system with $K = 1$ and $t = p = \left\lfloor \frac{M}{d+2} \right\rfloor$ (decimation).
4) Take the final approximation to be

$$\widetilde{f} = \widetilde{\Phi}\left(\{\widetilde{a}_{\ell,j}, \widetilde{x}_j\}\right) + \sum_{|k| \le M} \left\{ c_k(f) - \frac{1}{2\pi} \sum_{j=1}^{K} \mathrm{e}^{-\imath \widetilde{x}_j k} \sum_{\ell=0}^{d} \frac{\widetilde{a}_{\ell,j}}{(\imath k)^{\ell+1}} \right\} \mathrm{e}^{\imath k x}.$$

$\{N, 2N, \dots, M\}$. As before, $|\epsilon| \sim M^{-1}$, but now due to decimation we get accuracy $|\Delta x_j| \sim N^{-d-1} N^{-1} \sim M^{-d-2}$. In [3], [7] we develop an algorithm (see Algorithm 1) which in fact attains this accuracy. This result can be summarized as follows.

Theorem 4. *Let $f \in PC(d, K)$, so that $f = \Phi^{(d)} + \Psi$ where $\Phi^{(d)}$ is the piecewise polynomial with Fourier coefficients* (5), *and $\Psi \in C^d$. Assume that there exist constants J, A, B, R such that*

$$\begin{aligned} \min_{i \ne j} |x_i - x_j| &\ge J > 0, \quad |c_k(\Psi)| \le R \cdot k^{-d-2}, \\ |a_{\ell,j}| &\le A < \infty, \quad |a_{0,j}| \ge B > 0. \end{aligned}$$

Then the approximation $\widetilde{f}$ obtained by Algorithm 1 satisfies for $M \gg 1$

$$\begin{aligned} |\widetilde{x}_j - x_j| &\le C_1(d, K, J, A, B, R) \cdot M^{-d-2}; \\ |\widetilde{a}_{\ell,j} - a_{\ell,j}| &\le C_2(d, K, J, A, B, R) \cdot M^{\ell-d-1}, \quad 0 \leqslant \ell \leqslant d; \\ \left|\widetilde{f}(x) - f(x)\right| &\le C_3(d, K, J, A, B, R) \cdot M^{-d-1}. \end{aligned}$$

Note that the pointwise bound $\left|f(x) - \widetilde{f}(x)\right|$ is valid "away from discontinuities". Some numerical experiments, elaborated in [3], [7], confirm these theoretical accuracy predictions.

IV. Future work

Stable solution of Prony-type systems in the most general setting must take into account the possibility of colliding nodes. We believe that a reparametrization of the equations in the basis of finite differences is a promising approach to this problem. We have obtained initial results in [5], [20], and plan to continue in this direction.

The Fourier inversion problem for piecewise-analytic functions is still widely open (see e.g. [1]). While our results provide spectral convergence in this setting, it is still unknown if the algebraic method can be pushed to exponential or at least root-exponential accuracy.

Edge detection from spectral data is a well-researched problem, see e.g. [14], [15] and references therein. We expect that the 1D procedure can be generalized to treat the general case via some form of a "separation", or "slice reconstruction" (see e.g. [4] for an example of such a method, dealing with reconstruction from moments).

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